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SHAPES OF INSCRIBED AND CIRCUMSCRIBED QUADRANGLES

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A shape of quadrangle is an ordered pair of complex numbers corresponding to an orbit of quadrangles under the group of the plane direct similarities. We apply the shape for examination of the properties of inscribed and circumscribed quadrangles, a quadrangle with mutually perpendicular diagonals and a deltoid. Generalizations for inscribed polygons also are considered.

Using a complex cross-ratio for studying of the Euclidean plane is a well- known method considered in many books (see [3], [7] and [8]). This method is developed by J. A. Lester. She created systematically a complex analytic formalism for examining of triangle geometry (see [4], [5] and [6]). The shape of triangle is a fundamental tool in this formalism. The concept of the shape was extended by R. Artzy in [1]. He introduces a shape of polygon and proves some theorems for shapes of quadrangles. In this paper, we apply shapes for studing of inscribed quadrangles and polygons, circumscribed quadrangles and quadrangles with mutually perpendicular diagonals.

First, we recall some basic definitions. Details can be found in [1], [2] and [4]. We suppose that every point in the Euclidean plane is uniquely determined by a complex number, i.e. we consider the Gaussian plane. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be four noncollinear points in the Gaussian plane. The shape of the ordered triangle $\triangle\mathbf{abc}$ is the complex number $\dfrac{\mathbf{c}-\mathbf{a}}{\mathbf{b}-\mathbf{a}}$, the shape of the ordered quadrangle $\mathbf{abcd}$ is the pair of the complex numbers $[p,q]$, where p and q are the shapes of the triangles $\triangle\mathbf{abc}$ and $\triangle\mathbf{acd}$. Similarly, the shape of ordered convex polygon $\mathbf{z}_1\mathbf{z}_2\ldots\mathbf{z}_n$ is the ordered (n-2)-tuple $[p_2,p_3,\ldots,p_{n-1}]$, where

$$p_j=(\mathbf{z}_{j+1}-\mathbf{z}_1)(\mathbf{z}_j-\mathbf{z}_1)^{-1},\quad j=2,3,\ldots,n-1.$$

It is clear that the shape of triangle (quadrangle) is corresponded to the equivalence class of triangles (quadrangles) with respect to the direct similarities.

Theorem 1. *Let* $\mathbf{abcd}$ *be a quadrangle with shape* $[p,q]$ *and let* $\mathbf{k}$*,* $\mathbf{l}$*,* $\mathbf{m}$*,* $\mathbf{n}$ *be centroids of* $\triangle\mathbf{abc}$, $\triangle\mathbf{bcd}$, $\triangle\mathbf{cda}$, $\triangle\mathbf{dab}$, *respectively. Then the shape of the quadrangle* $\mathbf{klmn}$ *is equal to* $[1-(pq)^{-1},(1-q)p(1-pq)^{-1}]$.

Proof. Let the shape of the quadrangle $\mathbf{klmn}$ be $[p',q']$. We have that $\mathbf{k}=\dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$, $\mathbf{l}=\dfrac{\mathbf{b}+\mathbf{c}+\mathbf{d}}{3}$, $\mathbf{m}=\dfrac{\mathbf{c}+\mathbf{d}+\mathbf{a}}{3}$, $\mathbf{n}=\dfrac{\mathbf{d}+\mathbf{a}+\mathbf{b}}{3}$. Then

$$(1)\qquad p'=\frac{\mathbf{m}-\mathbf{k}}{\mathbf{l}-\mathbf{k}}=\frac{\mathbf{d}-\mathbf{b}}{\mathbf{d}-\mathbf{a}}\quad\text{and}\quad q'=\frac{\mathbf{n}-\mathbf{k}}{\mathbf{m}-\mathbf{k}}=\frac{\mathbf{d}-\mathbf{c}}{\mathbf{d}-\mathbf{b}}.$$

From $p = \dfrac{\mathbf{c}-\mathbf{a}}{\mathbf{b}-\mathbf{a}}$ and $q = \dfrac{\mathbf{d}-\mathbf{a}}{\mathbf{c}-\mathbf{a}}$, it follows that $\mathbf{b} = p^{-1}(\mathbf{c}-\mathbf{a})+\mathbf{a}$ and $\mathbf{d} = q(\mathbf{c}-\mathbf{a})+\mathbf{a}$. Replacing in (1), we obtain $p' = 1-(pq)^{-1}$ and $q' = (1-q)p(1-pq)^{-1}$. Thus the proof is completed.

Let **abcd** be a quadrangle. In the following five propositions, we will always denote the centroid of $\triangle\mathbf{abc}$ by **k**, the centroid of $\triangle\mathbf{bcd}$ by **l**, the centroid of $\triangle\mathbf{cda}$ by **m** and the centroid of $\triangle\mathbf{dab}$ by **n**.

Proposition 1. *The quadrangle* **abcd** *can be inscribed in a circle if and only if the quadrangle* **klmn** *can be inscribed in a circle.*

Proof. A necessary and sufficient condition for a quadrangle **abcd** with shape $[p,q]$ to be inscribed in a circle is $|\arg(1-p)^{-1}|+|\arg(1-q^{-1})| = \pi$ (see [1], Theorem 7). Since the shape of **klmn** is $[1-(pq)^{-1}, (1-q)p(1-pq)^{-1}]$, the quadrangle **klmn** can be inscribed in a circle whenever

$$|\arg\{1-(1-\frac{1}{pq})\}^{-1}|+|\arg\{1-\frac{1-pq}{(1-q)p}\}| = \pi. \tag{2}$$

We simplify the left hand side and obtain $|\arg pq|+|\arg\dfrac{1-p^{-1}}{1-q}| = \pi$. From $p = \dfrac{\mathbf{c}-\mathbf{a}}{\mathbf{b}-\mathbf{a}}$ and $q = \dfrac{\mathbf{d}-\mathbf{a}}{\mathbf{c}-\mathbf{a}}$, we have $|\arg pq|+|\arg\dfrac{1-p^{-1}}{1-q}| = |\arg\dfrac{\mathbf{d}-\mathbf{a}}{\mathbf{b}-\mathbf{a}}|+|\arg\dfrac{\mathbf{b}-\mathbf{c}}{\mathbf{d}-\mathbf{c}}| = \pi$. Then the equality (2) becomes $|\arg\triangle_{\mathbf{abd}}|+|\arg\triangle_{\mathbf{cdb}}| = \pi$ or $|\angle\mathbf{bad}|+|\angle\mathbf{dcb}| = \pi$, i.e. the quadrangle **klmn** can be inscribed in a circle whenever **abcd** can be inscribed (see Figure 1a).

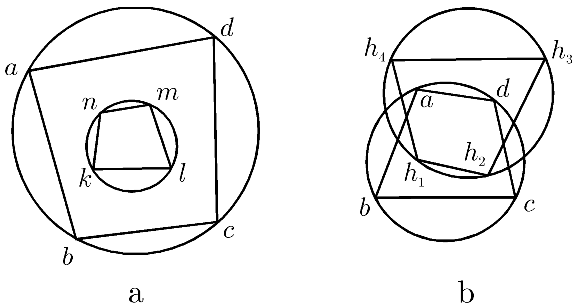

Figure 1

Proposition 2. *A circle can be inscribed in the quadrangle* **abcd** *if and only if a circle can be inscribed in the quadrangle* **klmn**.

Proof. A necessary and sufficient condition for a circle to be inscribed in the quadrangle **abcd** with shape $[p,q]$ is $|p^{-1}|+|1-q| = |1-p^{-1}|+|q|$ (see [1], Theorem 8). Then a circle can be inscribed in **klmn** if and only if

$$|\frac{1}{1-(pq)^{-1}}|+|1-\frac{(1-q)p}{1-pq}| = |1-\frac{1}{1-(pq)^{-1}}|+|\frac{(1-q)p}{1-pq}|. \tag{3}$$

We simplify the equality (3). Then it becomes $|q|+|1-p^{-1}| = |p^{-1}|+|q-1|$, i.e. a circle can be inscribed in the quadrangle **klmn** whenever a circle can be inscribed in **abcd**. Q.E.D.

Similarly, it is nearly to prove the following proposition:

Proposition 3. *The quadrangle* **abcd** *is*
(i) *a trapezoid if and only if the quadrangle* **klmn** *is a trapezoid,*
(ii) *a parallelogram (a rhombus, a rectangle, a square) if and only if the quadrangle* **klmn** *is a parallelogram (a rhombus, a rectangle, a square).*

Theorem 2. *Let* **abcd** *be a convex quadrangle with shape* $[p, q]$. *Its diagonals are perpendicular if and only if* $p^{-1} - q$ *is pure imaginary.*

Proof. The diagonals of the quadrangle **abcd** are perpendicular whenever $\frac{\mathbf{d}-\mathbf{b}}{\mathbf{a}-\mathbf{c}}$ is pure imaginary. Since $\mathbf{b} = p^{-1}(\mathbf{c}-\mathbf{a}) + \mathbf{a}$ and $\mathbf{d} = q(\mathbf{c}-\mathbf{a}) + \mathbf{a}$ (see Theorem 1) we obtain $\frac{\mathbf{d}-\mathbf{b}}{\mathbf{a}-\mathbf{c}} = p^{-1} - q$. This completes the proof.

By this Theorem and by Theorem 7 from [1] we observe that the diagonals **ac** and **bd** of a quadrangle **abcd** with shape $[p, q]$ are perpendicular chords in a circle if and only if the following two conditions are satisfied:
(i) $|arg(1-p)^{-1}| + |arg(1-q^{-1})| = \pi$,
(ii) $p^{-1} - q$ is pure imaginary.

Theorem 3. *A convex quadrangle* **abcd** *with shape* $[p, q]$ *is a deltoid if and only if* $p^{-1} = \overline{q}$.

Proof. Let **s** be a midpoint of **db**. Then a quadrangle **abcd** is a deltoid whenever its diagonals are perpendicular and **a**, **c** , **s** are collinear, i.e. $p^{-1} - q$ is imaginary and $\frac{\mathbf{a}-\mathbf{s}}{\mathbf{c}-\mathbf{s}}$ is real. Hence,

$$\frac{\mathbf{a}-\mathbf{s}}{\mathbf{c}-\mathbf{s}} = \frac{2\mathbf{a}-(\mathbf{d}+\mathbf{b})}{2\mathbf{c}-(\mathbf{d}+\mathbf{b})}. \tag{4}$$

From Theorem 1 we have $\mathbf{b} = p^{-1}(\mathbf{c}-\mathbf{a}) + \mathbf{a}$ and $\mathbf{d} = q(\mathbf{c}-\mathbf{a}) + \mathbf{a}$. Replacing in (4), we obtain $\frac{\mathbf{a}-\mathbf{s}}{\mathbf{c}-\mathbf{s}} = \frac{p^{-1}+q}{p^{-1}+q-2}$, i.e. $\frac{\mathbf{a}-\mathbf{s}}{\mathbf{c}-\mathbf{s}}$ is real whenever $p^{-1} + q$ is real. Hence, **abcd** is deltoid whenever $p^{-1} - q$ is imaginary and $p^{-1} + q$ is real. But $p^{-1} - q$ is imaginary if only if

$$p^{-1} - q = \overline{q} - \overline{p^{-1}} \tag{5}$$

and $p^{-1} + q$ is real if only if

$$p^{-1} + q = \overline{p^{-1}} + \overline{q}. \tag{6}$$

From equations (5) and (6) we obtain $p^{-1} = \overline{q}$.
Conversely, let $p^{-1} = \overline{q}$, i.e. $p\overline{q} = 1$. From $\triangle_{\mathbf{abd}} = \frac{\mathbf{a}-\mathbf{d}}{\mathbf{a}-\mathbf{b}} = \frac{\mathbf{a}-\mathbf{c}}{\mathbf{a}-\mathbf{b}}.\frac{\mathbf{a}-\mathbf{d}}{\mathbf{a}-\mathbf{c}} = pq$ it follows that $|\triangle_{\mathbf{abd}}| = |pq| = |p\overline{q}| = 1$. Since

$$\triangle_{\mathbf{cdb}} = \frac{\mathbf{c}-\mathbf{b}}{\mathbf{c}-\mathbf{d}} = \frac{\mathbf{c}-\mathbf{a}}{\mathbf{c}-\mathbf{d}}.\frac{\mathbf{c}-\mathbf{b}}{\mathbf{c}-\mathbf{a}} = \triangle_{\mathbf{cda}}.\triangle_{\mathbf{cab}} = (1-q)^{-1}.(1-p^{-1}),$$

then $|\triangle_{\mathbf{cdb}}| = |(1-q)^{-1}|.|(1-p^{-1})| = |1-q|^{-1}.|1-\overline{q}| = |1-q|^{-1}.|1-q| = 1$. This means that both triangles $\triangle$**abd** and $\triangle$**cdb** are isosceles with apeces at **a** and **c**, respectively. Then the quadrangle **abcd** is a deltoid. Q.E.D.

Proposition 4. *The diagonals of the quadrangle* **abcd** *are perpendicular if and only* *if the diagonals of* **klmn** *are perpendicular.*

Proof. From Theorem 2 the diagonals of **abcd** with shape $[p, q]$ are perpendicular whenever $p^{-1} - q$ is pure imaginary. The shape of **klmn** is $[1-(pq)^{-1}, (1-q)p(1-pq)^{-1}]$. Thus, the diagonals of **klmn** are perpendicular if and only if the number

$$\frac{1}{1-(pq)^{-1}} - \frac{(1-q)p}{1-pq} \tag{7}$$

is pure imaginary. We simplify (7) and get $\dfrac{q}{q-p^{-1}} - \dfrac{q-1}{q-p^{-1}} = \dfrac{1}{q-p^{-1}}$. Hence the expression (7) is pure imaginary whenever $p^{-1} - q$ is pure imaginary. Q.E.D.

Proposition 5. *A convex quadrangle* **abcd** *is a deltoid* ($|\mathbf{a}-\mathbf{b}| = |\mathbf{a}-\mathbf{d}|$ *and* ($|\mathbf{c}-\mathbf{b}| = |\mathbf{c}-\mathbf{d}|$) *if only if the quadrangle* **lmnk** *is a deltoid.*

Proof. The quadrangle **lmnk** with shape $[p', q']$ is a deltoid if and only if $p'^{-1} = \overline{q'}$. From the proof of Theorem 1., we get $p' = \dfrac{\mathbf{l}-\mathbf{n}}{\mathbf{l}-\mathbf{m}} = \dfrac{\mathbf{c}-\mathbf{a}}{\mathbf{b}-\mathbf{a}} = p$ and $q' = \dfrac{\mathbf{l}-\mathbf{k}}{\mathbf{l}-\mathbf{n}} = \dfrac{\mathbf{d}-\mathbf{a}}{\mathbf{c}-\mathbf{a}}$ (see Figure 2). Consequently, the quadrangle **lmnk** is a deltoid if and only if $p^{-1} = \overline{q}$ and this completes the proof.

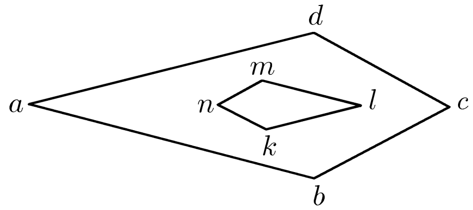

Figure 2

From equations (1), it follows that the shape of **klmn** is equal to the shape of a quadrangle **dabc**, i.e. the quadrangles **klmn** and **dabc** are similar and have the same orientation. Hence, all previous propositions are particular cases of Theorem 1.

Proposition 6. *Let* **abcd** *be a quadrangle inscribed in a circle with shape* $[p, q]$. *Then the ordered quadrangle with vertices in the orthocentres of the triangles* $\triangle\mathbf{abc}$, $\triangle\mathbf{bcd}$, $\triangle\mathbf{cda}$ *and* $\triangle\mathbf{dab}$ *has a shape* $[1-(pq)^{-1}, (1-q)p(1-pq)^{-1}]$ *and it can be also inscribed in a circle.*

Proof. Let $\mathbf{h_1}$ be a orthocentre of the triangle $\triangle\mathbf{abc}, \mathbf{h_2}$ of $\triangle\mathbf{bcd}, \mathbf{h_3}$ of $\triangle\mathbf{cda}$ and $\mathbf{h_4}$ of $\triangle\mathbf{dab}$ (see Figure 1b). Let **o** be a circumcentre of **abcd**. Then we have the equalities $\mathbf{h_1} = \mathbf{a}+\mathbf{b}+\mathbf{c}-2.\mathbf{o}$, $\mathbf{h_2} = \mathbf{b}+\mathbf{c}+\mathbf{d}-2.\mathbf{o}$, $\mathbf{h_3} = \mathbf{c}+\mathbf{d}+\mathbf{a}-2.\mathbf{o}$, $\mathbf{h_4} = \mathbf{d}+\mathbf{a}+\mathbf{b}-2.\mathbf{o}$. The shape of the ordered quadrangle $\mathbf{h_1h_2h_3h_4}$ is $[(\mathbf{h_3}-\mathbf{h_1})(\mathbf{h_2}-\mathbf{h_1})^{-1}, (\mathbf{h_4}-\mathbf{h_1})(\mathbf{h_3}-\mathbf{h_1})^{-1}]$. Replacing, we obtain that $(\mathbf{h_3}-\mathbf{h_1})(\mathbf{h_2}-\mathbf{h_1})^{-1} = (\mathbf{d}-\mathbf{b})(\mathbf{d}-\mathbf{a})^{-1}$ and $(\mathbf{h_4}-\mathbf{h_1})(\mathbf{h_3}-\mathbf{h_1})^{-1} = (\mathbf{d}-\mathbf{c})(\mathbf{d}-\mathbf{b})^{-1}$. From $p = \triangle_{\mathbf{abc}}$ and $q = \triangle_{\mathbf{acd}}$, it follows that $\mathbf{a}-\mathbf{b} = p^{-1}(\mathbf{a}-\mathbf{c})$ and $\mathbf{a}-\mathbf{d} = q(\mathbf{a}-\mathbf{c})$. Subtracting the last equalities we have that $\mathbf{d}-\mathbf{b} = (p^{-1}-q)(\mathbf{a}-\mathbf{c})$ whence we obtain $(\mathbf{d}-\mathbf{b})(\mathbf{d}-\mathbf{a})^{-1} = 1-(pq)^{-1}$. Since $\triangle_{\mathbf{cda}} = (1-q)^{-1}$ then $\mathbf{d}-\mathbf{c} = (1-q)(\mathbf{a}-\mathbf{c})$ and $(\mathbf{d}-\mathbf{c})(\mathbf{d}-\mathbf{b})^{-1} = (1-q)p(1-pq)^{-1}$. The second assertion follows from Theorem 7 in [1] for the ordered quadrangle **dabc** with shape $[1-(pq)^{-1}, (1-q)p(1-pq)^{-1}]$.

Corollary 6.1. *Let* **abcd** *be a quadrangle inscribed in a circle. Then a circle can be inscribed in the ordered quadrangle with vertices in the orthocentres of the triangles* $\triangle$**abc**, $\triangle$**bcd**, $\triangle$**cda** *and* $\triangle$**dab** *if and only if a circle can be inscribed in the quadrangle* **abcd**.

Proof. Let $[p, q]$ be a shape of the ordered quadrangle **dabc**. From Proposition 6 it follows that the ordered quadrangle with vertices in the orthocentres of the triangles $\triangle$**abc**, $\triangle$**bcd**, $\triangle$**cda** and $\triangle$**dab** and the quadrangle **dabc** have the same shapes. Using Theorem 8 from [1] we have that $|p^{-1}| + |1 - q| = |1 - p^{-1}| + |q|$ and this completes the proof.

Corollary 6.2. *Let* **abcd** *be a quadrangle inscribed in a circle. Then the ordered quadrangle with vertices in the orthocentres of the triangles* $\triangle$**abc**, $\triangle$**bcd**, $\triangle$**cda** *and* $\triangle$**dab** *is a trapezoid if and only if the quadrangle* **abcd** *is a trapezoid.*

Proof. Let $[p, q]$ be the shape of the ordered quadrangle **abcd**.
Then from [1, Theorem 6] it follows that **abcd** is a trapezoid if and only if $p = r(1-q)^{-1}$ for some nonzero real r such that sgn r = sgn (Im p). Since the ordered quadrangle with vertices in the orthocentres of the triangles $\triangle$**abc**, $\triangle$**bcd**, $\triangle$**cda** and $\triangle$**dab** and the quadrangle **dabc** have the same shapes, the quadrangle of the orthocenters is also trapezoid.

Corollary 6.3. *Let* **abcd** *be a quadrangle inscribed in a circle. Then the ordered quadrangle with vertices in the centroids of the triangles* $\triangle$**abc**, $\triangle$**bcd**, $\triangle$**cda** *and* $\triangle$**dab** *and the ordered quadrangle with vertices in the orthocentres of the same triangles are similar and have the same orientation.*

The proof follows from Theorem 1 and Proposition 6.

Theorem 4. *An ordered n-gon* $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ *with shape* $[p_2, \ldots, p_{n-1}]$ *can be inscribed in a circle if and only if*

$$|arg(1 - p_2 p_3 \ldots p_j)^{-1}| + |arg(1 - p_{j+1}^{-1})| = \pi \quad for \quad j = 2, \ldots, n-2.$$

Proof. Set $\prod_{i=2}^{j} p_i = Q_j$, then $Q_j = (\mathbf{z}_{j+1} - \mathbf{z}_1)(\mathbf{z}_2 - \mathbf{z}_1)^{-1}$. An ordered n-gon $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ can be inscribed in a circle if and only if the ordered quadrangles $(\mathbf{z}_1,\ \mathbf{z}_2,\ \mathbf{z}_{j+1},\ \mathbf{z}_{j+2})$ for $j = 2, \ldots, n-2$ can be inscribed in a circle. Using Theorem 7 from [1] we obtain the necessary and sufficient conditions

$$|arg(1 - (\mathbf{z}_{j+1} - \mathbf{z}_1)(\mathbf{z}_2 - \mathbf{z}_1))^{-1}| + |arg(1 - p_{j+1}^{-1})| = \pi, i.e.$$

$$|arg(1 - p_2 p_3 \ldots p_j)^{-1}| + |arg(1 - p_{j+1}^{-1})| = \pi \quad for \quad j = 2, \ldots, n-2. \text{ Q.E.D.}$$

Theorem 5. *Let* $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ *be an n-gon. Let* $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n$ *be the centroids of the ordered* (n-1) *- gons* $(\mathbf{z}_1, \ldots, \mathbf{z}_{n-1})$; $(\mathbf{z}_2, \ldots, \mathbf{z}_n)$; $\ldots$; $(\mathbf{z}_n, \mathbf{z}_1, \ldots, \mathbf{z}_{n-2})$, *respectively. Then*
(i) *the n-gon* $(\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n)$ *is convex if and only if the n-gon* $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ *is convex,*
(ii) *the n-gon* $(\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n)$ *is regular if and only if the n-gon* $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ *is regular,*
(iii) *the n-gon* $(\mathbf{m}_1,\ \mathbf{m}_2,\ \ldots,\ \mathbf{m}_n)$ *is inscribed in a circle if and only if the n-gon* $(\mathbf{z}_1,\ \ldots\ ,\ \mathbf{z}_n)$ *is inscribed in a circle.*

Proof. From $\mathbf{m}_1 = \dfrac{\mathbf{z}_1 + \ldots + \mathbf{z}_{n-1}}{n-1}$, $\mathbf{m}_2 = \dfrac{\mathbf{z}_2 + \ldots + \mathbf{z}_n}{n-1}, \ldots,$

$\mathbf{m}_k = \dfrac{\mathbf{z}_k + \ldots + \mathbf{z}_n + \mathbf{z}_1 + \ldots + \mathbf{z}_{k-2}}{n-1}, \ldots,\ \mathbf{m}_n = \dfrac{\mathbf{z}_n + \mathbf{z}_1 + \ldots + \mathbf{z}_{n-2}}{n-1}$ we obtain that the shape of the n-gon $(\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n)$ is the ordered (n-2)-tuple

$$[(\mathbf{m}_3-\mathbf{m}_1)(\mathbf{m}_2-\mathbf{m}_1)^{-1}, \ldots, (\mathbf{m}_{k+1}-\mathbf{m}_1)(\mathbf{m}_k-\mathbf{m}_1)^{-1}, \ldots, (\mathbf{m}_n-\mathbf{m}_1)(\mathbf{m}_{n-1}-\mathbf{m}_1)^{-1}] =$$

$$= [(\mathbf{z}_n - \mathbf{z}_2)\ (\mathbf{z}_n - \mathbf{z}_1)^{-1}, \ldots,\ (\mathbf{z}_n - \mathbf{z}_k)\ (\mathbf{z}_n - \mathbf{z}_{k-1})^{-1}, \ldots, (\mathbf{z}_n - \mathbf{z}_{n-1})\ (\mathbf{z}_n - \mathbf{z}_{n-2})^{-1}]$$

which is the shape of the n-gon $(\mathbf{z}_n,\ \mathbf{z}_1, \ldots,\ \mathbf{z}_{n-1})$. Then the first assertion follows from [1, Theorem 3], the second assertion follows from [1, Theorem 5] and the third assertion follows from the above Theorem 4 .

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ШЕЙПОВЕ НА ВПИСАНИ И ОПИСАНИ ЧЕТИРИЪГЪЛНИЦИ

Георги Христов Георгиев, Радостина Петрова Енчева,
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Шейп (или форма) на четириъгълник е наредена двойка от комплексни числа отговарящ на орбитата на четириъгълника под действието на групата на всички запазващи ориентацията равнинни подобности. Ние прилагаме шейпа за изучаване на свойствата на вписани и описани четириъгълници, четириъгълник с взаимно перпендикулярни диагонали и делтоид. Обобщения за вписан многоъгълник също са разгледани.