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NECESSARY AND SUFFICIENT CONDITION FOR PSEUDOCONVEXITY OF A FUNCTION

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A first order characterization of pseudoconvex and of strictly pseudoconvex functions on $\mathbb{R}^n$ is given. A relationship with invexity is analysed.

1. Introduction. Throughout this work X is an open convex subset of the finite-dimensional Euclidean space $\mathbb{R}^n$.

Recall the following concepts.

The function $f : X \to \mathbb{R}$ is said to be *quasiconvex* iff

$$f(x + t(y - x)) \le \max\{f(x), f(y)\} \quad \text{whenever} \quad x, y \in X \quad \text{and} \quad 0 \le t \le 1.$$

Lemma 1.1. **([1])** *Let $f : X \to \mathbb{R}$ be a differentiable function. Then f is quasiconvex if and only if the following implication holds:*

$$x, y \in X,\ f(y) \le f(x) \quad \textit{imply} \quad \langle f'(x), y - x\rangle \le 0.$$

Here by $\langle\cdot,\cdot\rangle$ is denoted the usual scalar product in finite-dimensional space.

The differentiable function $f : X \to \mathbb{R}$ is said to be *pseudoconvex* (*strictly pseudoconvex*) on X [3] iff

$$x, y \in X,\ f(y) < f(x)\ (x \ne y,\ f(y) \le f(x)) \quad \text{imply} \quad \langle f'(x), y - x\rangle < 0.$$

This notion is introduced by Mangasarian in 1965.

The differentiable function $f : X \to \mathbb{R}$ is said to be *invex* on X if there exists a mapping $\eta : X \times X \to \mathbb{R}^n$ such that

$$f(y) - f(x) \ge \langle f'(x), \eta\rangle \quad \text{for all} \quad x, y \in X.$$

This notion is introduced by Hanson [2].

It is well-known that each pseudoconvex (strictly pseudoconvex) function is invex. Then one can put the following question: Is the mapping η from the definition of invexity of a special form if the invex function f is pseudoconvex. The answer is: $\eta(x, y) = p(x, y)(x - y)$, where $p(x, y)$ is some real positive function. The main result of the paper is a first order necessary and sufficient condition for pseudoconvexity and for strictly pseudoconvexity of a differentiable function, connected with this question.

2. A characterization of the pseudoconvex functions. This section contains only the following theorem:

Theorem 2.1. *The differentiable function* $f : X \to \mathbb{R}$ *is pseudoconvex (strictly pseudoconvex) if and only if there exists a positive function* $p : X \times X \to \mathbb{R}$ *such that*

$$(1) \qquad f(y) - f(x) \geq p(x,y)\,\langle f'(x), y - x\rangle \quad \text{for all} \quad x, y \in X$$

$$(\ f(y) - f(x) > p(x,y)\,\langle f'(x), y - x\rangle \quad \text{for all} \quad x, y \in X,\ x \neq y\ ).$$

Proof. We shall consider only the pseudoconvex case. The strictly pseudoconvex case is similar.

Sufficiency is obvious.

Necessity. Let f be pseudoconvex, but (1) fails. Consequently, there exist $x, y \in X$ satisfying

$$(2) \qquad f(y) - f(x) < p\,\langle f'(x), y - x\rangle \quad \text{for all} \quad p > 0.$$

By taking the limits, as $p \to 0$ we get that $f(y) \leq f(x)$. If $f(y) < f(x)$, then according to the pseudoconvexity $\langle f'(x), y - x\rangle < 0$. As a result, (2) must not be satisfied for all sufficiently large $p > 0$. If $f(y) = f(x)$, since each pseudoconvex function defined on an open convex set is quasiconvex, then by quasiconvexity, $\langle f'(x), y - x\rangle \leq 0$. The last inequality is a contradiction to the implication (2). □

The class of functions which satisfies the inequality (1) is called by Weir [4] strongly pseudoconvex. It follows from the proved theorem that this class coincides with the pseudoconvex ones.

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НЕОБХОДИМО И ДОСТАТЪЧНО УСЛОВИЕ ЗА ПСЕВДОИЗПЪКНАЛОСТ НА ФУНКЦИЯ

Всеволод Иванов Иванов

В статията е дадена характеризация от първи ред на псевдоизпъкналите и на строго псевдоизпъкналие функции в пространството $\mathbb{R}^n$. Анализирана е връзка с инвексните функции.