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EINSTEIN MANIFOLDS WITH POINT-WISE CONSTANT CHARACTERISTICAL COEFFICIENTS OF THE CURVATURE OPERATOR*

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The skew-symmetric curvature operator has been used for characterizing the 4-dimensional Einstein manifolds with the property: both characteristical coefficients of the curvature operator are point-wise constant. We classify these manifolds according to the representation of the curvature tensor.

1. Introduction. The skew-symmetric curvature operator κ has a special role in Riemannian geometry. It has been considered by G. Stanilov and R. Ivanova in 1994 ([5]). Later many authors have applied the operator κ to the study of the Riemannian and Einstein manifolds ([1], [3-11], [13]). G. Stanilov introduced a higher order generalization of the skew-symmetric curvature operator in Riemannian geometry ([15]). R. Ivanova and T. Kawaguchi found out an application at the operator κ also to the Information geometry ([12]). In our study we have considered the Riemannian manifolds which have point-wise constant characteristical coefficients of the operator κ ([5], [7], [12], [13]). Later the curvature tensors of this kind of manifolds has been called IP curvature tensors ([2]). Systems of necessary and sufficient conditions for a manifold to have the above mentioned property have been given in [8-9]. Using κ the 4-dimensional Einstein manifolds have been characterized in [6, 8, 10, 11]. We gave a classification of some manifolds with point-wise constant characteristical coefficients of the operator κ in [6]. Later this classification has been completed for $m = 4$ by S. Ivanov, I. Petrova ([4]) and for $m \geq 5$, $m \neq 7$ by P. Gilkey, J. Leahy, H. Sadofski ([2]) and P. Gilkey ([1]). Some progress in studying the case $m = 7$ has been made by P. Gilkey and U. Semmelman ([3]). T. Zhang generalized these results to the pseudo-Riemannian setting ([16]).

2. Preliminaries. Let (M, g) be a 4-dimensional Riemannian manifold and κ_{E^2} be the skew-symmetric curvature operator defined in [8] by

$$\kappa_{E^2} = \kappa_{X,Y}(u) = R(X,Y)u, \tag{2.1}$$

where R is the curvature tensor, X, Y is an arbitrary orthonormal pair of vectors in the tangent space M_p at a point $p \in M$, E^2 is the 2-dimensional tangent subspace of M_p spaned by the vectors X, Y and $u \in M_p$.

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We consider the 4-dimensional Einstein manifolds satisfying the property: both characteristical coefficients of the curvature operator κ_{E^2} are point-wise constant. The curvature tensor of these manifolds can be represented only by the metric tensor of the manifold:

Theorem 2.1 ([13]) *Let (M, g) be a 4-dimensional Einstein manifold and (e_1,e_2,e_3,e_4) be a Singer-Thorpe base of the tangent space M_p. Then the curvature tensor of the manifold has the following representation:*

$$R(x, y, z, u) = \sigma_{12}K_{12} + \sigma_{13}K_{13} + \sigma_{14}K_{14} + d_{1234}R_{1234} + d_{1342}R_{1342} + d_{1423}R_{1423}, \tag{2.2}$$

where

$$\begin{aligned}
\sigma_{12} &= \sigma_{12}(x, y, z, u)\\
&= [g(x, e_1)g(y, e_2) - g(x, e_2)g(y, e_1)][g(z, e_1)g(u, e_2) - g(z, e_2)g(u, e_1)]\\
&+ [g(x, e_3)g(y, e_4) - g(x, e_4)g(y, e_3)][g(z, e_3)g(u, e_4) - g(z, e_4)g(u, e_3)],\\
\sigma_{13} &= \sigma_{13}(x, y, z, u)\\
&= [g(x, e_1)g(y, e_3) - g(x, e_3)g(y, e_1)][g(z, e_1)g(u, e_3) - g(z, e_3)g(u, e_1)]\\
&+ [g(x, e_2)g(y, e_4) - g(x, e_4)g(y, e_2)][g(z, e_2)g(u, e_4) - g(z, e_4)g(u, e_2)],\\
\sigma_{14} &= \sigma_{14}(x, y, z, u)\\
&= [g(x, e_1)g(y, e_4) - g(x, e_4)g(y, e_1)][g(z, e_1)g(u, e_4) - g(z, e_4)g(u, e_1)]\\
&+ [g(x, e_2)g(y, e_3) - g(x, e_3)g(y, e_2)][g(z, e_2)g(u, e_3) - g(z, e_3)g(u, e_2)],
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
d_{1234} &= d_{1234}(x, y, z, u)\\
&= [g(x, e_1)g(y, e_2) - g(x, e_2)g(y, e_1)][g(z, e_3)g(u, e_4) - g(z, e_4)g(u, e_3)]\\
&+ [g(x, e_3)g(y, e_4) - g(x, e_4)g(y, e_3)][g(z, e_1)g(u, e_2) - g(z, e_2)g(u, e_1)],\\
d_{1342} &= d_{1342}(x, y, z, u)\\
&= [g(x, e_1)g(y, e_3) - g(x, e_3)g(y, e_1)][g(z, e_4)g(u, e_2) - g(z, e_2)g(u, e_4)]\\
&+ [g(x, e_4)g(y, e_2) - g(x, e_2)g(y, e_4)][g(z, e_1)g(u, e_3) - g(z, e_3)g(u, e_1)],\\
d_{1432} &= d_{1423}(x, y, z, u)\\
&= [g(x, e_1)g(y, e_4) - g(x, e_4)g(y, e_1)][g(z, e_2)g(u, e_3) - g(z, e_3)g(u, e_2)]\\
&+ [g(x, e_2)g(y, e_3) - g(x, e_3)g(y, e_2)][g(z, e_1)g(u, e_4) - g(z, e_4)g(u, e_1)].
\end{aligned} \tag{2.4}$$

All the tensors $\sigma_{12}(x, y, z, u), \sigma_{13}(x, y, z, u), \sigma_{14}(x, y, z, u)$ and $d_{1234}(x, y, z, u)$, $d_{1342}(x, y, z, u)$, $d_{1423}(x, y, z, u)$ included in the representation (2.2) of the curvature tensor are tensors of type (0,4) and have the following properties:

Proposition 2.1. ([13]) *The tensors $\sigma_{12}(x, y, z, u)$, $\sigma_{13}(x, y, z, u)$ and $\sigma_{14}(x, y, z, u)$ defined by* (2.3) *satisfy the equation:*

$$\sigma_{12}(x, y, z, u) + \sigma_{13}(x, y, z, u) + \sigma_{14}(x, y, z, u) = \pi(x, y, z, u), \tag{2.5}$$

where $\pi(x, y, zu,) = g(x, z)g(y, u) - g(x, u)g(y, z)$.

Proposition 2.2. ([13]) *The tensors* $d_{1234}(x,y,z,u)$*,* $d_{1342}(x,y,z,u)$ *and* $d_{1423}(x,y,z,u)$ *defined by* (2.4) *satisfy the equation:*

$$d_{1234}(x,y,z,u)+d_{1342}(x,y,z,u)+d_{1423}(x,y,z,u)=D(x,y,z,u), \tag{2.6}$$

where

$$D(x,y,z,u)=\begin{vmatrix} x^1 & x^2 & x^3 & x^4 \\ y^1 & y^2 & y^3 & y^4 \\ z^1 & z^2 & z^3 & z^4 \\ u^1 & u^2 & u^3 & u^4 \end{vmatrix}.$$

On the other hand, we have proved

Theorem 2.2. ([13]) *Let* (M,g) *be a* 4*-dimensional Einstein manifold and* κ_{E^2} *be the curvature operator defined by* (2.1)*. If* (M,g) *has point-wise constant characteristical coefficients of the operator* κ_{E^2}*, then it is of one of the following types:*

$$\begin{array}{ll}
1.) & K_{12}=K_{13}=K_{14}, \qquad R_{1234}=R_{1342}=R_{1423}=0,\\
2.) & K_{12}=-K_{13}=-K_{14}, \qquad R_{1234}=R_{1342}=R_{1423}=0,\\
3.) & K_{12}=-K_{13}=K_{14}, \qquad R_{1234}=R_{1342}=R_{1423}=0,\\
4.) & K_{12}=K_{13}=-K_{14}, \qquad R_{1234}=R_{1342}=R_{1423}=0,\\
5.) & K_{12}=R_{1342}=-R_{1423}, \qquad K_{13}=K_{14}=R_{1234}=0,\\
6.) & K_{12}=-R_{1342}=R_{1423}, \qquad K_{13}=K_{14}=R_{1234}=0,\\
7.) & K_{13}=-R_{1234}=R_{1423}, \qquad K_{12}=K_{14}=R_{1342}=0,\\
8.) & K_{12}=R_{1234}=-R_{1423}, \qquad K_{12}=K_{14}=R_{1342}=0,\\
9.) & K_{14}=R_{1234}=-R_{1342}, \qquad K_{12}=K_{13}=R_{1423}=0,\\
10.) & K_{14}=-R_{1234}=R_{1342}, \qquad K_{12}=K_{13}=R_{1423}=0,\\
11.) & K_{12}=-\frac{1}{2}K_{13}=-\frac{1}{2}K_{14}=\frac{1}{2}R_{1234}=-R_{1342}=-R_{1423},\\
12.) & K_{12}=-\frac{1}{2}K_{13}=-\frac{1}{2}K_{14}=-\frac{1}{2}R_{1234}=R_{1342}=R_{1423},\\
13.) & K_{12}=-2K_{13}=K_{14}=2R_{1324}=-R_{1342}=2R_{1423},\\
14.) & K_{12}=-2K_{13}=K_{14}=-2R_{1324}=R_{1342}=-2R_{1423},\\
15.) & K_{12}=K_{13}=-2K_{14}=2R_{1234}=2R_{1342}=-R_{1423},\\
16.) & K_{12}=K_{13}=-2K_{14}=-2R_{1234}=-2R_{1342}=R_{1423}.
\end{array} \tag{2.7}$$

3. A classification of the manifolds with point-wise constant characteristical coefficients of the operator κ_{E^2}**.** Let (M,g) be a 4-dimensional Einstein manifold with point-wise constant characteristical coefficients of the operator κ_{E^2} and (e_1,e_2,e_3,e_4) be a Singer-Thorpe base of M_p. Then the curvature tensor of M has the representation (2.2) and the components of the curvature tensor satisfy someone of the relations (2.7).

Using both the representation (2.2) and the relations (2.7), we can get a new representation of the curvature tensor in each of the 16 cases possible. In case 1) we can see from (2.7) that the following relations hold:

$$K_{12} = K_{13} = K_{14} = a, \qquad R_{1234} = R_{1342} = R_{1423} = 0. \tag{3.1}$$

Applying (3.1) and (2.5) to the representation (2.2), we get

$$R(x, y, z, u) = a[\sigma_{12}(x, y, z, u) + \sigma_{13}(x, y, z, u) + \sigma_{14}(x, y, z, u)] = a\pi(x, y, z, u),$$

i.e. in this case we have a manifold of constant sectional curvature a.

In case 2) we have $-K_{12} = K_{13} = K_{14} = a, \quad R_{1234} = R_{1342} = R_{1423} = 0$ and then

$$\begin{aligned} R(x, y, z, u) &= a[-\sigma_{12}(x, y, z, u) + \sigma_{13}(x, y, z, u) + \sigma_{14}(x, y, z, u)] \\ &= K_{12}[2\sigma_{12}(x, y, z, u) - \pi(x, y, z, u)]. \end{aligned}$$

Analogously to the case 2), in the next two cases we obtain:

$$\begin{aligned} &3)\ R(x, y, z, u) = K_{13}[2\sigma_{13}(x, y, z, u) - \pi(x, y, z, u)], \\ &4)\ R(x, y, z, u) = K_{14}[2\sigma_{14}(x, y, z, u) - \pi(x, y, z, u)]. \end{aligned}$$

In case 5) the relations are $K_{12} = R_{1342} = -R_{1423} = a, \quad K_{13} = K_{14} = R_{1234} = 0$. Then we get

$$R(x, y, z, y) = K_{12}[\sigma_{12}(x, y, z, u) + d_{1342}(x, y, z, u) - d_{1423}(x, y, z, u)].$$

The curvature tensors of the manifolds in the next five cases 6) - 10) have similar representations:

$$\begin{aligned} &6)\ R(x, y, z, u) = K_{12}[\sigma_{12}(x, y, z, u) - d_{1342}(x, y, z, u) + d_{1423}(x, y, z, u)] \\ &7)\ R(x, y, z, u) = K_{13}[\sigma_{13}(x, y, z, u) + d_{1423}(x, y, z, u) - d_{1234}(x, y, z, u)] \\ &8)\ R(x, y, z, u) = K_{13}[\sigma_{13}(x, y, z, u) - d_{1423}(x, y, z, u) + d_{1234}(x, y, z, u)] \\ &9)\ R(x, y, z, u) = K_{14}[\sigma_{14}(x, y, z, u) + d_{1234}(x, y, z, u) - d_{1342}(x, y, z, u)] \\ &10)\ R(x, y, z, u) = K_{14}[\sigma_{14}(x, y, z, u) - d_{1234}(x, y, z, u) + d_{1342}(x, y, z, u)]. \end{aligned}$$

In case 11) we have $K_{12} = -R_{1342} = -R_{1423} = a, \quad -K_{13} = -K_{14} = R_{1234} = 2a$ and

$$\begin{aligned} R(x, y, z, u) &= a[\sigma_{12}(x, y, z, u) - 2\sigma_{13}(x, y, z, y) - 2\sigma_{14}(x, y, z, u) \\ &+2d_{1234}(x, y, z, u) - d_{1342}(x, y, z, u) - d_{1423}(x, y, z, u)], \end{aligned}$$

which, via (2.5) and (2.6), yields

$$R(x, y, z, u) = K_{12}[3\sigma_{12}(x, y, z, u) - 2\pi(x, y, z, u) + 3d_{1234}(x, y, z, u) - D(x, y, z, u)].$$

The representation of $R(x, y, z, u)$ is similar when the manifold is of some of the next five types12) - 16):

$$\begin{aligned} &12)\ R(x, y, z, u) = K_{12}[3\sigma_{12}(x, y, z, u) - 2\pi(x, y, z, u) - 3d_{1234}(x, y, z, u) + D(x, y, z, u)] \\ &13)\ R(x, y, z, u) = K_{13}[3\sigma_{13}(x, y, z, u) - 2\pi(x, y, z, u) + 3d_{1342}(x, y, z, u) - D(x, y, z, u)] \\ &14)\ R(x, y, z, u) = K_{13}[3\sigma_{13}(x, y, z, u) - 2\pi(x, y, z, u) - 3d_{1342}(x, y, z, u) + D(x, y, z, u)] \\ &15)\ R(x, y, z, u) = K_{14}[3\sigma_{14}(x, y, z, u) - 2\pi(x, y, z, u) + 3d_{1423}(x, y, z, u) - D(x, y, z, u)] \\ &16)\ R(x, y, z, u) = K_{14}[3\sigma_{14}(x, y, z, u) - 2\pi(x, y, z, u) - 3d_{1423}(x, y, z, u) + D(x, y, z, u)]. \end{aligned}$$

Since instead of the base $(e_1,\ e_2,\ e_3,\ e_4)$ of the tangent space M_p we could consider the base $(-e_1,\ e_2,\ e_3,\ e_4)$ and since (2.5),(2.6) hold, we can deduce that the cases 6), 8), 10), 12), 14), 16) are equivalent to the cases 5), 7), 9), 11), 13), 15), respectively. Thus, we obtain that the curvature tensors of the manifolds with point-wise constant characteristical coefficients of the operator κ_{E^2} are of the following types:

$$
\begin{array}{ll}
 & 1) R(x,y,z,u)=a\pi(x,y,z,u) \\
 & 2) R(x,y,z,u)=K_{12}[2\sigma_{12}(x,y,z,u)-\pi(x,y,z,u)] \\
 & 3) R(x,y,z,u)=K_{13}[2\sigma_{13}(x,y,z,u)-\pi(x,y,z,u)] \\
 & 4) R(x,y,z,u)=K_{14}[2\sigma_{14}(x,y,z,u)-\pi(x,y,z,u)] \\
 & 5) R(x,y,z,u)=K_{12}[\sigma_{12}(x,y,z,u)+d_{1342}(x,y,z,u)-d_{1423}(x,y,z,u)] \\
(3.2) & 6) R(x,y,z,u)=K_{13}[\sigma_{13}(x,y,z,u)+d_{1423}(x,y,z,u)-d_{1234}(x,y,z,u)] \\
 & 7) R(x,y,z,u)=K_{14}[\sigma_{14}(x,y,z,u)+d_{1234}(x,y,z,u)-d_{1342}(x,y,z,u)] \\
 & 8) R(x,y,z,u)=K_{12}[3\sigma_{12}(x,y,z,u)+3d_{1234}(x,y,z,u)-2\pi(x,y,z,u)-D(x,y,z,u)] \\
 & 9) R(x,y,z,u)=K_{13}[3\sigma_{13}(x,y,z,u)+3d_{1342}(x,y,z,u)-2\pi(x,y,z,u)-D(x,y,z,u)] \\
 & 10) R(x,y,z,u)=K_{14}[3\sigma_{14}(x,y,z,u)+3d_{1423}(x,y,z,u)-2\pi(x,y,z,u)-D(x,y,z,u)].
\end{array}
$$

Now one can easily notice that the cases 3) and 4) in the classification (3.2) can be obtained from the case 2) by a cyclic permutation of the indices $2 \to 3 \to 4$, i.e. both of the relations 3) and 4) are equivalent to the case 2). Analogously, the cases 6), 7) and 9), 10) are equivalent to the cases 5) and 8), respectively. In such a way, we have proved the following classification

Theorem 3.1. *Let (M,g) be a 4-dimensional Einstein manifold and κ_{E^2} be the curvature operator defined by* (2.1)*. If the characteristical coefficients of the operator κ_{E^2} are point-wise constant, then the curvature tensor of M is represented in one of the following ways:*

1) $R(x,y,z,u) = a\pi(x,y,z,u)$,
2) $R(x,y,z,u) = K_{12}[2\sigma_{12}(x,y,z,u) - \pi(x,y,z,u)]$,
3) $R(x,y,z,u) = K_{12}[\sigma_{12}(x,y,z,u) + d_{1342}(x,y,z,u) - d_{1423}(x,y,z,u)]$,
4) $R(x,y,z,u) = K_{12}[3\sigma_{12}(x,y,z,u) + 3d_{1234}(x,y,z,u) - 2\pi(x,y,z,u) - D(x,y,z,u)]$,

where a is the constant sectional curvature of the manifold and the tensors $\sigma_{1i}(x,y,z,u)$, $i = 2,3,4$, $d_{ijkl}(x,y,z,u)$, i,j,k,l - different, $\pi(x,y,z,u)$ and $D(x,y,z,u)$ are defined by (2.3), (2.4), (2.5) *and* (2.6)*, respectively.*

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АЙНЩАЙНОВИ МНОГООБРАЗИЯ С ТОЧКОВО-ПОСТОЯННИ ХАРАКТЕРИСТИЧНИ КОЕФИЦИЕНТИ НА ОПЕРАТОРА НА КРИВИНАТА

Райна Борисова Иванова

Косо-симетричният оператор на кривина беше използван за характеризиране на 4-мерните Айнщайнови многообразия със свойството: и двата характеристични коефициента на оператора на кривина са точково постоянни. Тези многообразия са класифицирани в съответствие с представянето на тензора на кривината.