МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2001
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2001
Proceedings of Thirtieth Spring Conference of
the Union of Bulgarian Mathematicians
Borovets, April 8–11, 2001

ADDITIVE MATRIX OPERATORS

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In this paper we consider properties of additive complex matrix operators which are not necessarily linear. In particular we define norms of such operators. These operators arise in the perturbation analysis of complex symmetric algebraic matrix equations, such as algebraic Lyapunov equations, Riccati equations, associated Riccati equations [1], etc. An application of the results obtained to the perturbation analysis of the standard continuous-time algebraic Lyapunov equation is demonstrated.

Keywords: Additive matrix operators, matrix equations, perturbation analysis.
AMS Subject classification: 15A04, 65F35

1. Introduction and notation. In this paper we consider some properties of complex additive matrix operators which may not be linear operators. We are especially interested in determining norms of complex additive operators.

We use the following notations: $\mathbb{R}$ and $\mathbb{C}$ – the sets of real and complex numbers, $\mathbb{R}_+ = [0, \infty)$; $\mathbb{F}$ – a replacement of $\mathbb{R}$ or $\mathbb{C}$; $\mathbb{F}^{m\times n}$ – the space of $m\times n$ matrices over $\mathbb{F}$; $A^\top$, $\overline{A}$ and $A^{\mathrm{H}} = \overline{A}^\top$ – the transpose, complex conjugate and complex conjugate transpose of the matrix A; A^* – the matrix $A^\top$ if A is real or the matrix A^{H} if A is complex; $\mathrm{vec}(A) \in \mathbb{F}^{mn}$ – the column-wise vector representation of $A \in \mathbb{F}^{m\times n}$; $\Pi_{mn} \in \mathbb{F}^{mn\times mn}$ – the vec-permutation matrix such that $\mathrm{vec}(A^\top) = \Pi_{mn}\mathrm{vec}(A)$; $\mathrm{spect}(A) \subset \mathbb{C}$ – the set of eigenvalues of $A \in \mathbb{C}^{n\times n}$; $\|\cdot\|_p$ – the Hölder p-norm ($1 \le p \in \mathbb{R}_+$) in $\mathbb{F}^n$; $\|A\|_{pq} = \max\{\|Ax\|_p : \|x\|_q = 1\}$ – the Hölder (p,q)-norm of $A \in \mathbb{F}^{m\times n}$, $\|A\|_p = \|A\|_{pp}$; $\|\cdot\|_{\mathrm{F}}$ – the Frobenius norm in $\mathbb{F}^{m\times n}$; $\mathbf{Lin}(p,m,n,q,\mathbb{F})$ – the set of linear matrix operators $\mathbb{F}^{m\times n} \to \mathbb{F}^{p\times q}$, $\mathbf{Lin}(m,n,\mathbb{F}) = \mathbf{Lin}(m,m,n,n,\mathbb{F})$, $\mathbf{Lin}(n,\mathbb{F}) = \mathbf{Lin}(n,n,n,n,\mathbb{F})$; $\mathbf{Lyap}(n,\mathbb{F})$ – the set of Lyapunov matrix operators $\mathcal{L} : \mathbb{F}^{n\times n} \to \mathbb{F}^{n\times n}$, defined by the condition $\mathcal{L}(X^*) = (\mathcal{L}(X))^*$ (we stress that $\mathbf{Lyap}(n,\mathbb{R}) \subset \mathbf{Lin}(n,\mathbb{R})$ but $\mathbf{Lyap}(n,\mathbb{C})$ is not a subspace of $\mathbf{Lin}(n,\mathbb{C})$); $\mathrm{Mat}(\mathcal{L}) \in \mathbb{F}^{pq\times mn}$ – the matrix of the operator $\mathcal{L} \in \mathbf{Lin}(p,m,n,q,\mathbb{F}$, defined via $\mathrm{vec}(\mathcal{L}(X)) = \mathrm{Mat}(\mathcal{L})\mathrm{vec}(X)$.

We also consider the vec-operator $\mathrm{vec}(\cdot)$ as a linear isomorphism between the corresponding linear spaces of matrices and vectors. Here the size of matrices may be skipped without any misunderstanding. However, the notation of the inverse vec-operator $\mathrm{vec}^{-1}(\cdot)$ may require an indication of the size of the resulting matrices. For example, given $l \in \mathbb{N}$ and $m,n \in \mathbb{N}$ with $mn = l$, the operator $\mathrm{vec}^{-1}_{m,n}(\cdot)$ transforms vectors from $\mathbb{F}^l$ into matrices from $\mathbb{F}^{m\times n}$. If the co-domain of the inverse vec-operator is clear from the context, we omit the subindexes in the notation $\mathrm{vec}^{-1}(\cdot)$.

The abbreviation ":=" means "equal by definition".

2. Motivating example. In this section we consider a complex Lyapunov matrix equation, arising in the stability analysis and other areas of systems theory. The perturbation analysis of such equations leads to additive operators which are non-linear pseudo-polynomial functions.

Consider the perturbation analysis problem for the complex continuous-time Lyapunov matrix algebraic equation

$$\mathcal{L}_A(X) := A^{\mathrm{H}}X + XA = C, \tag{1}$$

where $A, C \in \mathbb{C}^{n\times n}$ are given matrix coefficients and $X \in \mathbb{C}^{n\times n}$ is the unknown matrix. We suppose that the Lyapunov operator $\mathcal{L}_A \in \mathbf{Lyap}(n,\mathbb{C})$ is invertible which is equivalent to the requirement that for all $\lambda, \mu \in \mathrm{spect}(A)$ it is fulfilled $\lambda + \overline{\mu} \neq 0$. If the matrix C is Hermitian (i.e., if $C^{\mathrm{H}} = C$) then the unique solution X of (1) is also Hermitian.

Let δA and δC be perturbations in A and C. Denote $\delta_A := \|\delta A\|_{\mathrm{F}}$, $\delta_C := \|\delta C\|_{\mathrm{F}}$. We suppose that δ_A is small enough so that the perturbed Lyapunov operator $\mathcal{L}_{A+E}$ is invertible for all $E \in \mathbb{C}^{n\times n}$ with $\|E\|_{\mathrm{F}} \leq \delta_A$. In this case there exists a unique solution $Y = X + \delta X$ of the perturbed Lyapunov equation

$$\mathcal{L}_{A+\delta A}(Y) := (A+\delta A)^{\mathrm{H}}Y + Y(A+\delta A) = C + \delta C. \tag{2}$$

The aim of norm-wise perturbation analysis for equation (1) is to find a bound for the quantity $\delta_X := \|\delta X\|_{\mathrm{F}}$ as a function of the perturbation vector $\delta := [\delta_C, \delta_A]^{\top} \in \mathbb{R}^2_+$. A number of perturbation results in this area are known.

Consider first the local perturbation analysis in which the bound is of the form

$$\delta_X \leq \omega(\delta) + O(\|\delta\|^2),\ \delta \to 0,$$

where $\omega(\delta) = O(\|\delta\|)$, $\delta \to 0$. Here the function $\omega : \mathbb{R}^2_+ \to \mathbb{R}_+$ is first order homogeneous in the sense that $\omega(\alpha\delta) = \alpha\omega(\delta)$ for all $\alpha \in \mathbb{R}_+$.

Equation (2) may be rewritten as an operator equation for δX as

$$\delta X = \mathcal{L}_A^{-1}(\delta C) - \mathcal{L}_A^{-1}(\delta A^{\mathrm{H}}X + X\delta A) - G(\delta A, \delta X), \tag{3}$$

where $G(E,Z) = E^{\mathrm{H}}Z + ZE$. Keeping first order terms and taking the vec-operation from both sides of (3) we get

$$x = Lc + Ma + N\overline{a} + O(\|\delta\|^2),\ \delta \to 0.$$

Here $x := \mathrm{vec}(\delta X)$, $c := \mathrm{vec}(\delta C)$, $a := \mathrm{vec}(\delta A)$ and

$$L := \left(I_n \otimes A^{\mathrm{H}} + A^{\top} \otimes I_n\right)^{-1},\ M := -L(I_n \otimes X),\ N := -L(X \otimes I_n)\Pi_{n^2}.$$

Having in mind that $\delta_X = \|x\|_2$ we obtain

$$\delta_X \leq \omega(\delta) + O(\|\delta\|^2),\ \delta \to 0,$$

where

$$\omega(\delta) := \max\left\{\|Lc + Ma + N\overline{a}\|_2 : \|c\|_2 \leq \delta_C,\ \|a\|_2 \leq \delta_A\right\}.$$

Furthermore, if $2\|L\|_2\delta_A < 1$, then the perturbation bound for δ_X is

$$\delta_X \leq f(\delta) := \frac{\omega(\delta)}{1 - 2\|L\|_2\delta_A}.$$

Of course, having in mind that $\|M\|_2 = \|N\|_2$, we can bound $f_1(\delta)$ from above as

$$\omega(\delta) \leq \|L\|_2\delta_C + 2\|M\|_2\delta_A.$$

Unfortunately, this may give rather weak results.

The quantity $\omega(\delta)$ is obtained via an optimization procedure. It involves the operator $\mathcal{M} : \mathbb{C}^{2n\times 2n} \to \mathbb{C}^{2n\times 2n}$, defined by $\mathcal{M}(a) = Ma + N\overline{a}$. This operator is additive, $\mathcal{M}(a+b) = \mathcal{M}(a) + \mathcal{M}(b)$, but it is not homogeneous ($\mathcal{M}(\lambda a) \neq \lambda \mathcal{M}(a)$ if $0 \neq \lambda \notin \mathbb{R}$ and $0 \neq a \notin \mathbb{R}^{2n\times 2n}$). These means that $\mathcal{M}$ is not a linear operator over the field $\mathbb{C}$. However, we can obtain a new real operator, associated with $\mathcal{M}$ (this operator so called a realification of $\mathcal{M}$), which is linear over the field $\mathbb{R}$. Operators of this type are considered in the next section.

3. Main results. Let the matrix function (or matrix operator) of matrix argument

$$\mathcal{F} = [f_{ij}] : \mathbb{F}^{m\times n} \to \mathbb{F}^{p\times q}$$

be given, where $f_{ij} : \mathbb{F}^{m\times n} \to \mathbb{F}$ are scalar functions of matrix argument. By $\mathcal{F}^\top = [f_{ji}]$ and $\mathcal{F}^{\mathrm{H}} = [\overline{f}_{ji}]$ we denote the operators, transposed and complex conjugate transposed to the operator $\mathcal{F}$, respectively. We also set $\mathcal{F}^*$ for $\mathcal{F}^\top$ in the real case and for $\mathcal{F}^{\mathrm{H}}$ in the complex case.

Every matrix operator $\mathcal{F} : \mathbb{F}^{m\times n} \to \mathbb{F}^{p\times q}$ is equivalent to a vector function $f : \mathbb{F}^{mn} \to \mathbb{F}^{pq}$. Indeed, set

$$f(x) = \mathrm{vec}(\mathcal{F}(\mathrm{vec}^{-1}(x)),$$

where $x := \mathrm{vec}(X)$ and $X = \mathrm{vec}^{-1}(x)$.

A complex operator $\mathcal{F} : \mathbb{C}^{m\times n} \to \mathbb{C}^{p\times q}$ may be *realificated*, resulting in the equivalent operator

$$\mathcal{F}^{\mathbb{R}} : \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times n} \to \mathbb{R}^{p\times q} \times \mathbb{R}^{p\times q}$$

as follows Let $X = X_0 + \imath X_1 \in \mathbb{C}^{m\times n}$ and

$$\mathcal{F}(X) = \mathcal{F}_0(X_0, X_1) + \imath \mathcal{F}_1(X_0, X_1),$$

where $X_i \in \mathbb{R}^{m\times n}$ and $\mathcal{F}_i(X_0, X_1) \in \mathbb{R}^{p\times q}$. Then we may define the realified operator $\mathcal{F}^{\mathbb{R}}$ via

$$\mathcal{F}^{\mathbb{R}}(X_0, X_1) := (\mathcal{F}_0(X_0, X_1), \mathcal{F}_1(X_0, X_1)) \in \mathbb{R}^{p\times q} \times \mathbb{R}^{p\times q}.$$

Sometimes it is convenient to write the realification $\mathcal{F}^{\mathbb{R}}(X_0, X_1)$ of $\mathcal{F}(X)$ also as

$$\mathcal{F}^{\mathbb{R}}(X_0, X_1) := \begin{bmatrix} \mathcal{F}_0(X_0, X_1) \\ \mathcal{F}_1(X_0, X_1) \end{bmatrix} \in \mathbb{R}^{2p\times q}.$$

For X as above set

$$\begin{aligned} \mathrm{vec}^{\mathbb{R}}(X) &:= \begin{bmatrix} \mathrm{vec}(X_0) \\ \mathrm{vec}(X_1) \end{bmatrix} \in \mathbb{R}^{2mn}, \\ X^{\mathbb{R}} &:= \begin{bmatrix} X_0 & -X_1 \\ X_1 & X_0 \end{bmatrix} \in \mathbb{R}^{2m\times 2n}. \end{aligned}$$

Next we shall use the realifications

$$\begin{aligned} \mathrm{vec}^{\mathbb{R}}(AXB) &= (B^\top \otimes A)^{\mathbb{R}} \mathrm{vec}^{\mathbb{R}}(Z) \in \mathbb{R}^{2pq}, \\ (B^\top \otimes A)^{\mathbb{R}} &= \begin{bmatrix} B_0^\top \otimes A_0 - B_1^\top \otimes A_1 & -\left(B_1^\top \otimes A_0 + B_0^\top \otimes A_1\right) \\ B_1^\top \otimes A_0 + B_0^\top \otimes A_1 & B_0^\top \otimes A_0 - B_1^\top \otimes A_1 \end{bmatrix} \end{aligned}$$

and

$$\mathrm{vec}^{\mathbb{R}}(Az) = A^{\mathbb{R}} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} \in \mathbb{R}^{2p},$$

where $A \in \mathbb{C}^{p\times m}$, $X \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times q}$ $A_i \in \mathbb{R}^{p\times m}$, $B_i \in \mathbb{R}^{n\times q}$and $z = z_0 + \imath z_1 \in \mathbb{C}^m$;

$z_0, z_1 \in \mathbb{R}^m$. Hence, if $\mathcal{L} \in \mathbf{Lin}(p,m,n,q,\mathbb{C})$ and $\mathrm{Mat}(\mathcal{L}) \in \mathbb{C}^{pq\times mn}$ is the matrix representation of $\mathcal{L}$, then

$$\mathrm{vec}^{\mathbb{R}}(\mathcal{L}(X)) = \mathrm{Mat}^{\mathbb{R}}(\mathcal{L})\mathrm{vec}^{\mathbb{R}}(X).$$

Definition 3.1. *We recall* [2] *that the operator* $\mathcal{F} : \mathbb{F}^{m\times n} \to \mathbb{F}^{p\times q}$ *is* additive *if* $\mathcal{F}(X+Y) = \mathcal{F}(X) + \mathcal{F}(Y)$*,* homogeneous *if* $\mathcal{F}(\alpha X) = \alpha X$ *and* semi-homogeneous *if* $\mathcal{F}(\alpha X) = \overline{\alpha}\mathcal{F}(X)$ *for all* $X, Y \in \mathbb{F}^{m\times n}$ *and* $\alpha \in \mathbb{F}$*. The operator* $\mathcal{F}$ *is* linear *if it is additive and homogeneous,*

$$\mathcal{F}(\alpha X + \beta Y) = \alpha\mathcal{F}(X) + \beta\mathcal{F}(Y),$$

and semi-linear *if it is additive and semi-homogeneous,*

$$\mathcal{F}(\alpha X + \beta Y) = \overline{\alpha}\mathcal{F}(X) + \overline{\beta}\mathcal{F}(Y),$$

for all $X, Y \in \mathbb{F}^{m\times n}$ *and* $\alpha, \beta \in \mathbb{F}$*.*

In the real case $\mathbb{F} = \mathbb{R}$ the properties of linearity and semi-linearity coincide. Also, a complex semi-linear operator 'becomes' linear if we consider $\mathbb{C}^{n\times n}$ as a linear space over $\mathbb{R}$ instead of $\mathbb{C}$. This is based on the observation that a linear space V over any field $\mathbb{F}$ (including the space $V = \mathbb{F}$) is also a linear space over any subfield $\mathbb{E}$ of $\mathbb{F}$. If in particular $\mathbb{F}$ is a finite extension of $\mathbb{E}$ of degree $k = [\mathbb{F} : \mathbb{E}]$ then the dimension of V over $\mathbb{F}$ is k times the dimension of V over $\mathbb{E}$.

A general operator $\mathcal{L} \in \mathbf{Lin}(p,m,n,q,\mathbb{F})$ may be represented as

$$\mathcal{L}(X) = \sum_{i=1}^{r} A_i X B_i, \tag{4}$$

where $A_i \in \mathbb{F}^{p\times m}$, $B_i \in \mathbb{F}^{n\times q}$ are given matrix coefficients and r is the *Sylvester index* of $\mathcal{L}$, i.e., the minimum number of terms, required in the representation of $\mathcal{L}$ as a sum of elementary linear operators $X \mapsto A_i X B_i$, see [3]. The matrix of the operator $\mathcal{L}$ is

$$\mathrm{Mat}(\mathcal{L}) = \sum_{i=1}^{r} B_i^{\top} \otimes A_i \in \mathbb{F}^{pq\times mn}.$$

Similarly, a general semi-linear operator $\mathcal{M} : \mathbb{C}^{m\times n} \to \mathbb{C}^{p\times q}$ admits the representation

$$\mathcal{M}(X) = \mathcal{N}(X^{\mathrm{H}}) = \sum_{i=1}^{s} C_i X^{\mathrm{H}} D_i \tag{5}$$

with $C_i \in \mathbb{C}^{p\times n}$, $B_i \in \mathbb{C}^{m\times q}$, or

$$\mathcal{M}(X) = \mathcal{L}(\overline{X}) = \sum_{i=1}^{r} A_i \overline{X} B_i, \tag{6}$$

where $\mathcal{N} \in \mathbf{Lin}(p,n,m,q,\mathbb{C})$, $\mathcal{L} \in \mathbf{Lin}(p,m,n,q,\mathbb{C})$. The realifications of (5) and (6) are

$$\begin{aligned}
\mathrm{vec}^{\mathbb{R}}(\mathcal{M}(X)) &= \mathrm{Mat}^{\mathbb{R}}(\mathcal{N}) \begin{bmatrix} \Pi_{mn}\mathrm{vec}(X_0) \\ -\Pi_{mn}\mathrm{vec}(X_1) \end{bmatrix} \\
&= \mathrm{Mat}^{\mathbb{R}}(\mathcal{R})\mathrm{diag}\left(\Pi_{mn}, -\Pi_{mn}\right)\mathrm{vec}^{\mathbb{R}}(X)
\end{aligned}$$

and

$$\mathrm{vec}^{\mathbb{R}}(\mathcal{M}(X)) = \mathrm{Mat}^{\mathbb{R}}(\mathcal{N})\mathrm{diag}\left(I_{mn}, -I_{mn}\right)\mathrm{vec}^{\mathbb{R}}(X),$$

where $X = X_0 + \imath X_1 \in \mathbb{C}^{m\times n}$, $X_0, X_1 \in \mathbb{R}^{m\times n}$. Thus we may define the matrix of the realification $\mathcal{M}^{\mathbb{R}}$ of the semi-linear operator $\mathcal{M}$ as

$$\begin{aligned}\mathrm{Mat}(\mathcal{M}^{\mathbb{R}}) &= \mathrm{Mat}^{\mathbb{R}}(\mathcal{L})\mathrm{diag}\,(\Pi_{mn}, -\Pi_{mn}) \\ &= \mathrm{Mat}^{\mathbb{R}}(\mathcal{N})\mathrm{diag}\,(I_{mn}, -I_{mn})\,.\end{aligned}$$

Note that a semi-linear complex operator $\mathcal{F}$ is not differentiable. However, its realification $\mathcal{F}^{\mathbb{R}}$ is a linear operator. We note that if $\mathcal{F}$ is a linear operator, so is $\mathcal{F}^{\top}$, while $\mathcal{F}^{\mathrm{H}}$ is semi-linear.

Taking the vec-operation from both sides of the expressions (4) and (5) for a linear or a semi-linear operator we get

$$\mathrm{vec}(\mathcal{L}(X)) = L\mathrm{vec}(X)$$

and

$$\mathrm{vec}(\mathcal{M}(X)) = L\Pi_{mn}\mathrm{vec}(\overline{X}),$$

where

$$L := \mathrm{Mat}(\mathcal{L}) := \sum_{i=1}^{r} B_i^{\top} \otimes A_i \in \mathbb{F}^{pq\times mn}$$

is the matrix representation of the linear operator $\mathcal{L}$.

We shall also deal with complex additive operators $\mathcal{F}$, which may be represented as sum of a linear and a semi-linear operator, i.e.,

$$\mathcal{F}(X) = \mathcal{L}_1(X) + \mathcal{L}_2(X^{\mathrm{H}}),$$

where $\mathcal{L}_1 \in \mathbf{Lin}(p,m,n,q,\mathbb{C})$, $\mathcal{L}_2 \in \mathbf{Lin}(p,n,m,q,\mathbb{C})$. In this case we have

$$\mathrm{vec}^{\mathbb{R}}(\mathcal{F}(X)) = \Big(\mathrm{Mat}^{\mathbb{R}}(\mathcal{L}_1) + \mathrm{Mat}^{\mathbb{R}}(\mathcal{L}_2)\mathrm{diag}\,(\Pi_{nm}, -\Pi_{nm})\Big)\,\mathrm{vec}^{\mathbb{R}}(X).$$

Therefore we shall define the matrix of the realification $\mathcal{F}^{\mathbb{R}}$ of the additive operator $\mathcal{F}$ as

$$\mathrm{Mat}(\mathcal{F}^{\mathbb{R}}) := \mathrm{Mat}^{\mathbb{R}}(\mathcal{L}_1) + \mathrm{Mat}^{\mathbb{R}}(\mathcal{L}_2)\mathrm{diag}\,(\Pi_{nm}, -\Pi_{nm})\,.$$

Definition 3.2. *The operator $\mathcal{F} = [f_{ij}]$ is* polynomial *if its elements $f_{ij} : \mathbb{F}^{m\times n} \to \mathbb{F}$ are polynomial functions.*

A polynomial operator $\mathcal{F} : \mathbb{F}^{m\times n} \to \mathbb{F}^{p\times q}$ is globally Fréchet differentiable in the sense that for each $X_0 \in \mathbb{F}^{n\times n}$ we have

$$\mathcal{F}(X+Y) = \mathcal{F}(X) + \mathcal{L}(X,Y) + \mathcal{H}(X,Y),$$

where $\mathcal{L}(X,\cdot) \in \mathbf{Lin}(p,m,n,q,\mathbb{F})$ and

$$\lim_{Y\to 0} \frac{\|\mathcal{H}(X,Y)\|}{\|Y\|} = 0.$$

In this case the linear operator $\mathcal{L}(X,\cdot)$ is referred to as the *Fréchet derivative* of $\mathcal{F}$ at the point X and is denoted as $\mathcal{F}_X(X)(\cdot)$ or briefly as $\mathcal{F}_X(\cdot)$.

Definition 3.3. *The complex operator $\mathcal{F} : \mathbb{C}^{m\times n} \to \mathbb{C}^{p\times q}$ is said to be* pseudo-polynomial [1] *if it may be represented as*

$$\mathcal{F}(X) = \mathcal{G}(X, X^{\mathrm{H}}), \tag{7}$$

where $\mathcal{G} : \mathbb{F}^{m\times n} \times \mathbb{F}^{n\times m} \to \mathbb{F}^{p\times q}$ is a polynomial operator.

Pseudo-polynomial operators are not differentiable, but their realifications are real analytic operators. If $\mathcal{F}$ is the pseudo-polynomial operator, given by (7), we may define an additive operator $\widehat{\mathcal{F}}_X(\cdot)$ by

$$\widehat{\mathcal{F}}_X(Y) := \mathcal{G}_1(Y) + \mathcal{G}_2(Y^{\mathrm{H}}),$$

where $\mathcal{G}_k$ is the partial Fréchet derivative of $\mathcal{G}(X_1, X_2)$ in X_k, computed at $X_1 = X_0$, $X_2 = X_0^{\mathrm{H}}$. We have

$$\mathcal{F}(X+Y) = \mathcal{F}(X) + \widehat{\mathcal{F}}_X(Y) + \mathcal{H}(X,Y),$$

where $\mathcal{H}(X,Y) = o(\|Y\|)$, $Y \to 0$. Thus $\widehat{\mathcal{F}}_X(\cdot)$ is an analogue to the Fréchet derivative in case of pseudo-polynomial operators and is referred to as the *Fréchet pseudo-derivative* of $\mathcal{F}$ at the point X. We also denote the pseudo-derivative as $\mathcal{F}_X(\cdot)$. Whenever they exist, the Fréchet derivative and pseudo-derivative are unique.

If $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are Hölder or Frobenius norms in $\mathbb{F}^{p\times q}$ and $\mathbb{F}^{m\times n}$, respectively, then an *induced norm* of operators $\mathcal{L}$ from $\mathbf{Lin}(p,m,n,q,\mathbb{F})$ is defined as

$$\|\mathcal{L}\|_{\alpha,\beta} := \max\{\|\mathcal{L}(X)\|_\alpha : \|X\|_\beta = 1\}. \tag{8}$$

If the F-norm in $\mathbb{F}^{n\times n}$ is used, then

$$\begin{aligned} \|\mathcal{L}\|_{\mathrm{F}} &:= \max\{\|\mathcal{L}(X)\|_{\mathrm{F}} : \|X\|_{\mathrm{F}} = 1\} \\ &= \max\{\|\mathrm{vec}(\mathcal{L}(X))\|_2 : \|\mathrm{vec}(X)\|_2 = 1\} \\ &= \max\{\|\mathrm{Mat}(\mathcal{L})\mathrm{vec}(X)\|_2 : \|\mathrm{vec}(X)\|_2 = 1\} \\ &= \|\mathrm{Mat}(\mathcal{L})\|_2. \end{aligned} \tag{9}$$

When the operator $\mathcal{M}$ is semi-linear, e.g., $\mathcal{M}(X) = \mathcal{N}(X^{\mathrm{H}})$, $\mathcal{L} \in \mathbf{Lin}(p,n,m,q,\mathbb{C})$, we may again define its norm via (8) and (9) and thus the induced norm of $\mathcal{M}$ is equal to the induced norm of the underlying operator $\mathcal{L}$. However, if the complex operator $\mathcal{F}$ is only additive and not semi-linear or linear,

$$\mathcal{F}(X) = \mathcal{L}_1(X) + \mathcal{L}_2(X^{\mathrm{H}}),\ \mathcal{L}_1 \in \mathbf{Lin}(p,m,n,q,\mathbb{C}),\ \mathcal{L}_2 \in \mathbf{Lin}(p,n,m,q,\mathbb{C}), \tag{10}$$

then the determination of its induced norm is more subtle. Let

$$L_k = L_{k0} + \imath L_{k1} \in \mathbb{C}^{pq\times mn},\ k = 1,2,$$

be the matrix of the operator $\mathcal{L}_k$, where the matrices L_{kj} are real.

Definition 3.4. *The Frobenius norm of the additive operator $\mathcal{F}$, induced by the Frobenius norm in $\mathbb{C}^{m\times n}$ and $\mathbb{C}^{p\times q}$, is*

$$\|\mathcal{F}\|_{\mathrm{F}} := \max\{\|\mathcal{F}(X)\|_{\mathrm{F}} : \|X\|_{\mathrm{F}} \le 1\}.$$

We have

$$\|\mathcal{F}\|_{\mathrm{F}} = \max\{\|\mathrm{vec}(\mathcal{F}(X))\|_2 : \|\mathrm{vec}(X)\|_2 \le 1\}.$$

In view of the relations

$$\begin{aligned} \mathrm{vec}(\mathcal{F}(X)) &= \mathrm{vec}(\mathcal{L}_1(X)) + \mathrm{vec}(\mathcal{L}_2(X^{\mathrm{H}})) \\ &= L_1\mathrm{vec}(X) + L_2\Pi_{mn}\mathrm{vec}(\overline{X}) \end{aligned}$$

we obtain

$$\|\mathcal{F}\|_{\mathrm{F}} = \nu(L_1, L_2) := \|M(L_1,L_2)\|_2. \tag{11}$$

Here

$$M(L_1,L_2) := \mathrm{Mat}(\mathcal{F}^{\mathbb{R}}) = \begin{bmatrix} L_{10} + L_{20}\Pi_{mn} & -L_{11} + L_{21}\Pi_{mn} \\ L_{11} + L_{21}\Pi_{mn} & L_{10} - L_{20}\Pi_{mn} \end{bmatrix} \tag{12}$$

is the matrix of the realification $\mathcal{F}^{\mathbb{R}}$ of $\mathcal{F}$. Thus we have proved the following result.

Proposition 3.1. *The induced Frobenius norm of a complex additive operator $\mathcal{F}$ with a representation* (10) *is equal to the induced Frobenius norm of its realification defined via* (11) *and* (12).

The results obtained in this section find a wide application in the perturbation analysis of complex Lyapunov and Riccati equations. Particular details will be published elsewhere.

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АДИТИВНИ МАТРИЧНИ ОПЕРАТОРИ

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Разгледани са свойствата на една класа адитивни комплексни матрични оператори, които не са непременно линейни. В частност, дефинирани са норми на такива оператори. Тези оператори възникват при пертурбационния анализ на комплексни симетрични алгебрични матрични уравнения, например уравнения на Ляпунов, уравнения на Рикати, асоциирани уравнения на Рикати и др. Показано е приложението на получените резултати при пертурбационния анализ на стандартното непрекъснато уравнение на Ляпунов.