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A NOTE ON THE CONVERGENCE OF THE SOR BORSCH-SUPAN METHOD

Nikolay V. Kyurkchiev

Convergence properties of the SOR Borsch-Supan method for the simultaneous approximation of polynomial roots are considered. The choice of acceleration parameter is discussed.

1. Introduction. Let

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

be the polynomial with the simple or complex zeros x_i, $i = 1, \ldots, n$ and x_i^k, $i = 1, \ldots, n$ be distinct reasonably close approximations of these zeros. The method, which, in short, will always be referred to as the (BS)-method, (Borsch-Supan method) can be defined by the sequences

$$x_i^{k+1} = x_i^k - \frac{W_i(x_i^k)}{1 + \sum_{j\neq i}^{n} \frac{W_j(x_j^k)}{x_i^k - x_j^k}}, \; i = 1, \ldots, n, \; k = 0, 1, \ldots, \tag{1.1}$$

where x_i^{k+1} denotes the new approximation.

In this paper we are concerned with the successive overrelaxation Borsch-Supan method (SOR-BS method)

$$x_i^{k+1} = x_i^k - h_k \frac{W_i(x_i^k)}{1 + \sum_{j\neq i}^{n} \frac{W_j(x_j^k)}{x_i^k - x_j^k}}, \; i = 1, \ldots, n, \; k = 0, 1, \ldots, \tag{1.2}$$

where $W_i(x_i^k) = f(x_i^k)/\prod_{j\neq i}^{n}(x_i^k - x_j^k)$ and $h_k \in (0, 1]$ is an acceleration parameter. It should be noted that the SOR-BS nethod has a form of prediction-correction method. Let

$$\sigma_i^k = \sum_{j\neq i}^{n} \frac{W_j(x_j^k)}{x_i^k - x_j^k}, \; \max_{1\leq i\leq n} |W_i(x_i^k)| = W, \; \min_{j\neq i} |x_i^k - x_j^k| = d.$$

In practice, conditions for the convergence of the method (1.2) are given in the form of the inequality

$$h_k \frac{W}{|1+\sigma_i^k|} < \frac{\omega d}{2(n-1)}, \tag{1.3}$$

where $\omega > 0$ is a constant. The inequality (1.3) will be used in our analysis of the SOR-BS method. In later considerations we will find the upper bound for ω.

2. Main results. By the definition of d, W and σ_i^k we find

$$|1+\sigma_i^k| \geq 1 - |\sigma_i^k| > 1 - \frac{(n-1)W}{d} = \frac{d-(n-1)W}{d}.$$

We assume that

$$|x_i^{k+1} - x_i^k| \leq \frac{h_k W}{|1+\sigma_i^k|} \leq \frac{h_k W d}{d-(n-1)W} < \frac{\omega d}{2(n-1)},$$

i.e.

$$W < \frac{d}{2(n-1)}\beta_k, \tag{2.1}$$

where $\beta_k = \dfrac{\omega}{h_k + \frac{\omega}{2}}$.

Before establishing the convergence theorem for the method (1.2), we give some necessary estimates using the previous notations.

Lemma 2.1. *If the inequality* (2.1) *holds, then for* $i \in I_n$, *we have*

$$\begin{aligned}
&(i) \quad |\sigma_i^k| < \frac{\beta_k}{2}\\
&(ii) \quad \frac{2-\beta_k}{2} < |1+\sigma_i^k| > \frac{2+\beta_k}{2}\\
&(iii) \quad |x_i^{k+1} - x_j^k| > d\frac{2n-2-\omega}{2(n-1)}\\
&(iv) \quad |x_i^{k+1} - x_j^{k+1}| > d\frac{n-1-\omega}{n-1}\\
&(v) \quad \sum_{j\neq i}^{n} \frac{|W_j(x_j^k)|}{|x_i^{k+1} - x_j^k|} < \frac{\beta_k}{2-\omega}\\
&(vi) \quad \prod_{j\neq i}^{n} \left| \frac{x_i^{k+1} - x_j^k}{x_i^{k+1} - x_j^{k+1}} \right| < \left(1 + \frac{\frac{\omega}{2}}{n-1-\omega}\right)^{n-1} < e^{\frac{\omega}{2}}.
\end{aligned}$$

Proof. Of *(i):*

$$|\sigma_i^k| \leq \frac{(n-1)W}{d} < \frac{\beta_k}{2}.$$

Of *(ii):*

$$|1+\sigma_i^k| \leq 1 + |\sigma_i^k| < \frac{2+\beta_k}{2}, \quad |1+\sigma_i^k| \geq 1 - |\sigma_i^k| > \frac{2-\beta_k}{2}.$$

Of *(iii):*

$$|x_i^{k+1} - x_j^k| \geq |x_i^k - x_j^k| - |x_i^{k+1} - x_i^k| > d - \frac{\omega d}{2(n-1)} = d\frac{2n-2-\omega}{2(n-1)}.$$

Of *(iv):*

$$|x_i^{k+1} - x_j^{k+1}| \geq |x_i^k - x_j^k| - |x_i^{k+1} - x_i^k| - |x_j^{k+1} - x_j^k| > d - 2\frac{\omega d}{2(n-1)} = d\frac{n-1-\omega}{n-1}.$$

Of *(v):*

$$\sum_{j\neq i}^{n} \frac{|W_j(x_j^k)|}{|x_i^{k+1} - x_j^k|} \leq \frac{(n-1)W2(n-1)}{d(2n-2-\omega))} = \beta_k \frac{n-1}{2n-2-\omega} \leq \frac{\beta_k}{2-\omega}.$$

The last inequality $\frac{n-1}{2n-2-\omega} \leq \frac{1}{2-\omega}$ holds for each $n \geq 2$.
Of *(vi):*

$$\begin{aligned} \prod_{j\neq i}^{n} \left| \frac{x_i^{k+1} - x_j^k}{x_i^{k+1} - x_j^{k+1}} \right| &\leq \prod_{j\neq i}^{n} \left(1 + \frac{|x_j^{k+1} - x_j^k|}{|x_i^{k+1} - x_j^{k+1}|} \right) < \left(1 + \frac{\frac{\omega d}{2(n-1)}}{\frac{d(n-1-\omega)}{n-1}} \right)^{n-1} \\ &= \left(1 + \frac{\frac{\omega}{2}}{n-1-\omega} \right)^{n-1} < e^{\frac{\omega}{2}}. \end{aligned}$$

Using the estimates given in Lemma 2.1 we prove

Lemma 2.2. *If the inequality* (2.1) *holds, then*

$$|W_i(x_i^{k+1})| < \left(1 - h_k + \frac{\beta_k}{2} + \frac{\beta_k h_k}{2-\omega} \right) \frac{2}{2-\beta_k} e^{\frac{\omega}{2}} |W_i(x_i^k)|. \tag{2.2}$$

Proof. Evidently

$$f(x) = \prod_{j=1}^{n} (x - x_j^k) \left(1 + \frac{W_i(x_i^k)}{x - x_i^k} + \sum_{j\neq i}^{n} \frac{W_j(x_j^k)}{x - x_j^k} \right). \tag{2.3}$$

From (1.2) we obtain

$$\frac{W_i(x_i^k)}{x_i^{k+1} - x_i^k} = -\frac{1}{h_k}(1 + \sigma_i^k).$$

According to this, the relation (2.3) for $x = x_i^{k+1}$ becomes

$$f(x_i^{k+1}) = (x_i^{k+1} - x_i^k) \prod_{j\neq i}^{n} (x_i^{k+1} - x_j^k) \left(1 - \frac{1}{h_k}(1 + \sigma_i^k) + \sum_{j\neq i}^{n} \frac{W_j(x_j^k)}{x_i^{k+1} - x_j^k} \right).$$

After dividing with $\prod_{j\neq i}^{n} (x_i^{k+1} - x_j^{k+1})$ we obtain

$$W_i(x_i^{k+1}) = \frac{f(x_i^{k+1})}{\prod_{j\neq i}^{n}(x_i^{k+1}-x_j^{k+1})} = (x_i^{k+1}-x_i^k)\prod_{j\neq i}^{n}\frac{x_i^{k+1}-x_j^k}{x_i^{k+1}-x_j^{k+1}}\,A, \tag{2.4}$$

where

$$A = 1-\frac{1}{h_k}(1+\sigma_i^k)+\sum_{j\neq i}^{n}\frac{W_j(x_j^k)}{x_i^{k+1}-x_j^k}.$$

Starting from (2.4) and using estimates of Lemma 2.1 we find

$$A = \frac{h_k-1}{h_k}-\frac{\sigma_i^k}{h_k}+\sum_{j\neq i}^{n}\frac{W_j(x_j^k)}{x_i^{k+1}-x_j^k},$$

$$h_k|A| \le 1-h_k+|\sigma_i^k|+h_k\sum_{j\neq i}^{n}\frac{|W_j(x_j^k)|}{|x_i^{k+1}-x_j^k|} \le 1-h_k+\frac{\beta_k}{2}+\frac{h_k\beta_k}{2-\omega}.$$

$$\begin{aligned}|W_i(x_i^{k+1})| &\le |x_i^{k+1}-x_i^k||A|e^{\frac{\omega}{2}} \le h_k\frac{|W_(x_i^k)|}{1+\sigma_i^k|}|A|e^{\frac{\omega}{2}}\\ &< \left(1-h_k+\frac{\beta_k}{2}+\frac{\beta_k h_k}{2-\omega}\right)\frac{2}{2-\beta_k}e^{\frac{\omega}{2}}|W_i(x_i^k)|.\end{aligned}$$

We consider the contraction factor Q for the SOR-BS method appearing in the relation

$$|x_i^{k+2}-x_i^{k+1}| < Q|x_i^{k+1}-x_i^k|.$$

We assume that the ratio $\frac{h_{k+1}}{h_k}$ is very close to 1. We have the following theorem

Theorem 2.1. *For the contraction factor Q the following formula is valid*

$$Q = q(\omega, h_k)\alpha,$$

where

$$\begin{aligned}q(\omega,h_k) &= \left(1-h_k+\frac{\beta_k}{2}+\frac{\beta_k h_k}{2-\omega}\right)\frac{2(2+\beta_k)}{(2-\beta_k)^2}e^{\frac{\omega}{2}},\\ \alpha &= \frac{h_{k+1}}{h_k}.\end{aligned}$$

Proof. Let $c_i^t = x_i^t - x_i^{t+1},\ \ t = k, k+1$. Then

$$|c_i^k| = h_k\frac{|W_i(x_i^k)|}{|1+\sigma_i^k|} \le h_k|W_i(x_i^k)|\frac{2}{2-\beta_k}.$$

By induction

$$|c_i^{k+1}| = h_{k+1}\frac{|W_i(x_i^{k+1})|}{|1+\sigma_i^{k+1}|} \le h_{k+1}|W_i(x_i^{k+1})|\frac{2}{2-\beta_{k+1}}.$$

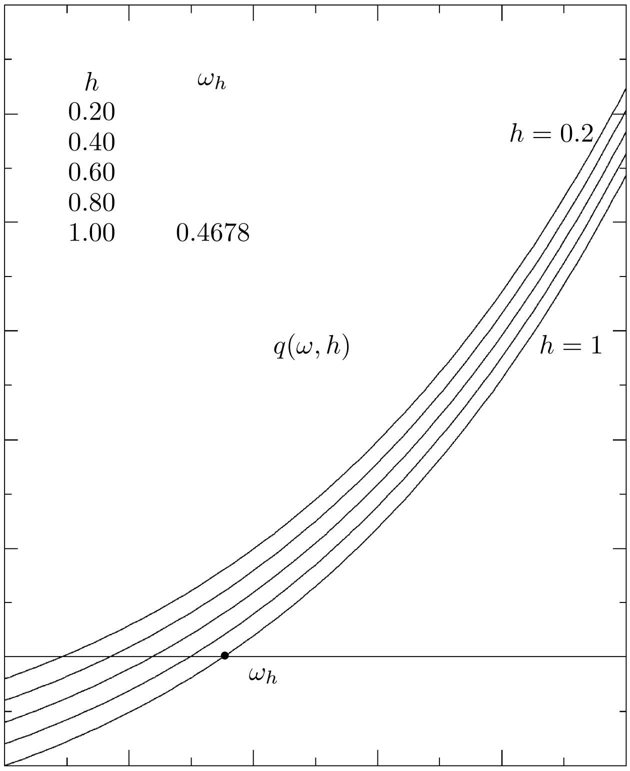

Fig. 1. Contraction of the Borsch-Supan SOR method

According to this and by (2.2), we obtain

$$
\begin{aligned}
|c_i^{k+1}| &\leq h_{k+1}\frac{2}{2-\beta_{k+1}}\frac{2}{2-\beta_k}\left(1-h_k+\frac{\beta_k}{2}+\frac{\beta_k h_k}{2-\omega}\right)e^{\frac{\omega}{2}}|W_i(x_i^k)|\frac{h_k}{h_k}\frac{|1+\sigma_i^k|}{|1+\sigma_i^k|} \\
&\leq \frac{h_{k+1}}{h_k}\frac{4}{(2-\beta_{k+1})(2-\beta_k)}\left(1-h_k+\frac{\beta_k}{2}+\frac{\beta_k h_k}{2-\omega}\right)e^{\frac{\omega}{2}}\frac{2+\beta_k}{2}|c_i^k|.
\end{aligned}
$$

Evidently $2-\beta_{k+1} > 2-\beta_k$ and

$$|x_i^{k+2}-x_i^{k+1}| < \alpha q(\omega, h_k)|x_i^{k+1}-x_i^k| = Q|x_i^{k+1}-x_i^k|.$$

This proves the theorem. □

According to the relation above in Fig. 1 we represent the quantity $q(\omega, h)$ as a function of ω, taking h as a parameter. The upper bound, corresponding to $q = 1$, is denoted by ω_h. The h_k increase to 1, defining BS-method in the continuation of the iteration procedure ($\alpha = 1$). By the way, from the equation

$$q(\omega, 1) = \frac{\omega(4-\omega)(1+\omega)}{2(2-\omega)} e^{\frac{\omega}{2}} = 1$$

we find the upper bound for $\omega: \ \omega < \omega_h \approx 0.4678$.

The optimal values of h_i^k (optimal in the sense that convergence is guaranteed) are not known. Other results have been found by Petkovic, Herceg, Ilic [2], Petkovic, Kyurkchiev [3], Kyurkchiev [1].

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Nikolay V. Kyurkchiev
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia, Bulgaria
e-mail: `nkyurk.math.bas.bg`

БЕЛЕЖКА ВЪРХУ СХОДИМОСТТА НА РЕЛАКСАЦИОННИЯ МЕТОД НА БЬОРСК-СУПАН

Николай В. Кюркчиев

В тази статия се третират въпроси свързани с избора на начални апроксимации гарантиращи сходимостта на релаксационния метод на Бьорск-Супан.