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A NOTE ON A SUBCLASS OF MEROMORPHIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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In this paper we obtain coefficient inequalities and distortion theorems for the class $T^*(\alpha,\beta,A,B)$ of meromorphic convex functions with negative coefficients.

1. Introduction and definitions. Let Σ denote the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n \tag{1}$$

which are analytic in the punctured unit disk

$$U^*=\{z:z\in C \text{ and } 0<|z|<1\}$$

with a simple pole at the origin with residue 1 there. Let Σ^* denote the subclass of Σ consisting of functions $f(z)$, which are convex with respect to the origin, that is, satisfying the condition

$$\operatorname{Re}\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>0,\qquad z\in U^*.$$

Let $\Sigma^*(\alpha)$ denote the subclass of Σ consisting of functions $f(z)$ which are convex of order α, that is, satisfying the condition

$$\operatorname{Re}\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\alpha,\quad z\in U^*,\quad 0\le\alpha<1.$$

Let $\Sigma^*(\alpha,A,B)$ denote the class of functions $f(z)$ in Σ which satisfy the condition that $\dfrac{(zf'(z))'}{f'(z)}$ has a representation of the form given by

$$1+\frac{zf''(z)}{f'(z)}=-\frac{1+\{B+(A-B)(1-\alpha)\}w(z)}{1+Bw(z)}. \tag{2}$$

Here $w(z)$ is analytic in U and satisfies the conditions

$$w(0)=0 \text{ and } |w(z)|<1,\quad z\in U$$

$(0 \le \alpha < 1,\ -1 \le A < B \le 1,\ 0 < B \le 1)$. The condition (2) is equivalent to

$$\left| \frac{\dfrac{zf''(z)}{f'(z)} + 2}{B\left(1 + \dfrac{zf''(z)}{f'(z)}\right) + [B + (A-B)(1-\alpha)]} \right| < 1, \quad z \in U^*.$$

We note also that

$$\Sigma^*(\alpha, -1, 1) = \Sigma^*(\alpha).$$

Let T denote the subclass of Σ consisting of functions of the form:

$$f(z) = \frac{1}{z} - \sum_{n=1}^{\infty} |a_n| z^n. \tag{3}$$

Definition. *A function $f(z)$ in Σ is in the class $\Sigma^*(\alpha, \beta, A, B)$ if it satisfies the condition*

$$\left| \frac{\dfrac{zf''(z)}{f'(z)} + 2}{B\left(1 + \dfrac{zf''(z)}{f'(z)}\right) + [B + (A-B)(1-\alpha)]} \right| < \beta$$

$(z \in U^*,\ 0 \le \alpha < 1,\ 0 < \beta \le 1,\ -1 \le A < B \le 1,\ 0 < B \le 1)$.

Let us write

$$T^*(\alpha, \beta, A, B) = \Sigma^*(\alpha, \beta, A, B) \cap T.$$

We note that $T^*(\alpha, \beta, A, 1)$ is the class of meromorphic convex functions with negative coefficients which was studied by the first author [1].

In this paper we obtain coefficient inequalities and distortion theorems for the class $T^*(\alpha, \beta, A, B)$. We employ techniques similar to those used by Silverman [2].

2. Coefficient inequalities.

Theorem 1. *Let the function $f(z)$ defined by* (1) *be analytic in U^*. If*

$$\sum_{n=1}^{\infty} \{(n+1) + \beta [Bn + (B-A)\alpha + A]\} n|a_n| \le (B-A)\beta(1-\alpha) \tag{4}$$

$(0 \le \alpha < 1,\ 0 < \beta \le 1,\ -1 \le A < B \le 1,\ 0 < B \le 1)$ *then*

$$f(z) \in \Sigma^*(\alpha, \beta, A, B).$$

Proof. Suppose that (4) holds true for all admissible values of α, β, A and B. Consider the expression

$$F(f, f') = |zf''(z) + 2f'(z)| - \beta |B\{f'(z) + zf''(z)\} + [B + (A-B)(1-\alpha)]f'(z)|. \tag{5}$$

Replacing f and f' by their series expansions, we have for $0 < |z| = r < 1$

$$\begin{aligned} F(f,f') &= \left|\sum_{n=1}^{\infty}(n+1)na_n z^{n-1}\right| - \\ &- \beta\left|\frac{(B-A)(1-\alpha)}{z^2} + \sum_{n=1}^{\infty}\left[Bn + (B-A)\alpha + A\right]na_n z^{n-1}\right| \end{aligned}$$

or

$$\begin{aligned} r^2F(f,f') &\le \sum_{n=1}^{\infty}(n+1)n\,|a_n|\,r^{n+1} - \beta\Bigg\{(B-A)(1-\alpha) - \\ &- \sum_{n=1}^{\infty}\left[Bn + (B-A)\alpha + A\right]n\,|a_n|\,r^{n+1}\Bigg\} = \\ &= \sum_{n=1}^{\infty}\left\{(n+1) + \beta\left[Bn + (B-A)\alpha + 1\right]n\,|a_n|\,r^{n+1}\right\} - \\ &- (B-A)\beta(1-\alpha). \end{aligned}$$

Since the above inequality holds true for all r $(0 < r < 1)$, by letting $r \to 1-$, we have

$$F(f,f') \le \sum_{n=1}^{\infty}\left\{(n+1) + \beta\left[Bn + (B-A)\alpha + A\right]\right\}n|a_n| - (B-A)\beta(1-\alpha) \le 0$$

by (4).

Hence it follows that

$$\left|\frac{zf''(z)}{f'(z)} + 2\right| < \beta\left|B\left(1 + \frac{zf''(z)}{f'(z)}\right) + \left[B + (A-B)(1-\alpha)\right]\right|$$

so that $f(z) \in \Sigma^*(\alpha,\beta,A,B)$. This completes the proof of Theorem 1. $\square$

Theorem 2. *Let the function $f(z)$ defined by* (3) *be analytic in U^*. Then $f(z) \in T^*(\alpha,\beta,A,B)$ if and only if* (4) *is satisfied.*

Proof. In view of Theorem 1, let us assume that the function $f(z)$, defined by (3), is in the class $T^*(\alpha,\beta,A,B)$. Then

$$\left|\frac{\dfrac{zf''(z)}{f'(z)} + 2}{B\left(1 + \dfrac{zf''(z)}{f'(z)}\right) + \left[B + (A-B)(1-\alpha)\right]}\right| =$$

$$= \left|\frac{-\displaystyle\sum_{n=1}^{\infty}(n+1)n|a_n|z^{n-1}}{\dfrac{(B-A)(1-\alpha)}{z^2} - \displaystyle\sum_{n=1}^{\infty}\left[Bn + (B-A)\alpha + A\right]n|a_n|z^{n-1}}\right| < \beta,\ z \in U^*.$$

Using the fact that $\operatorname{Re} z \le |z|$ for all z, we thus have

$$\text{(6)} \quad \operatorname{Re}\left\{ \frac{\sum_{n=1}^{\infty}(n+1)n|a_n|z^{n-1}}{\frac{(B-A)(1-\alpha)}{z^2} - \sum_{n=1}^{\infty}\left[Bn+(B-A)\alpha+A\right]n|a_n|z^{n-1}} \right\} < \beta,\ z \in U^*.$$

Now choose the values of z on the real axis so that

$$1 + \frac{zf''(z)}{f'(z)}$$

is real. By letting $z \to 1-$ through real values, we obtain

$$\sum_{n=1}^{\infty}(n+1)n|a_n| \le \beta\left\{(B-A)(1-\alpha) - \sum_{n=1}^{\infty}\left[Bn+(B-A)\alpha+A\right]n|a_n|\right\}$$

or

$$\text{(7)} \quad \sum_{n=1}^{\infty}\left\{(n+1)+\beta\left[Bn+(B-A)\alpha+A\right]\right\}n|a_n| \le (B-A)\beta(1-\alpha)$$

which proves Theorem 2. □

Corollary 1. *Let the function $f(z)$ defined by* (3) *be in the class $T^*(\alpha,\beta,A,B)$. Then*

$$|a_n| \le \frac{(B-A)\beta(1-\alpha)}{n\left\{(n+1)+\beta\left[Bn+(B-A)\alpha+A\right]\right\}}, \quad n \in \mathbb{N} := \{1,2,3,\dots\}$$

where equality holds true for functions of the form

$$f_n(z) = \frac{1}{z} - \frac{(B-A)\beta(1-\alpha)}{n\left\{(n+1)+\beta\left[Bn+(B-A)\alpha+A\right]\right\}}z^n, \quad n \in \mathbb{N}.$$

3. A distortion theorem.

Theorem 3. *Let the function f(z) defined by* (3) *be in the class $T^*(\alpha,\beta,A,B)$. Then, for $0 < |z| = r < 1$*

$$\text{(8)} \quad \frac{1}{r} - \frac{(B-A)\beta(1-\alpha)}{2+\beta\left[(B+A)+(B-A)\alpha\right]}r \le |f(z)| \le \frac{1}{r} + \frac{(B-A)\beta(1-\alpha)}{2+\beta\left[(B+A)+(B-A)\alpha\right]}r$$

where equality holds true for the function

$$\text{(9)} \quad f_1(z) = \frac{1}{z} - \frac{(B-A)\beta(1-\alpha)}{2+\beta\left[(B+A)+(B-A)\alpha\right]}z, \quad (z = ir, r)$$

and

$$\text{(10)} \quad \frac{1}{r^2} - \frac{(B-A)\beta(1-\alpha)}{2+\beta\left[(B+A)+(B-A)\alpha\right]} \le |f'(z)| \le \frac{1}{r^2} + \frac{(B-A)\beta(1-\alpha)}{2+\beta\left[(B+A)+(B-A)\alpha\right]}$$

where equality holds true for the function $f_1(z)$ *given by* (9) *at* $z = \pm r, \pm ir$.

Proof. In view of Theorem 2, we have

$$\sum_{n=1}^{\infty} |a_n| \leq \frac{(B-A)\beta(1-\alpha)}{2+\beta\left[(B+A)+(B-A)\alpha\right]}. \tag{11}$$

Thus, for $0 < |z| = r < 1$,

$$|f(z)| \leq \frac{1}{r} + \sum_{n=1}^{\infty} |a_n| r^n \leq \frac{1}{r} + r\sum_{n=1}^{\infty} |a_n| \leq \frac{1}{r} + \frac{(B-A)\beta(1-\alpha)}{2+\beta\left[(B+A)+(B-A)\alpha\right]} r \tag{12}$$

and

$$|f(z)| \geq \frac{1}{r} - \sum_{n=1}^{\infty} |a_n| r^n \geq \frac{1}{r} - r\sum_{n=1}^{\infty} |a_n| \geq \frac{1}{r} - \frac{(B-A)\beta(1-\alpha)}{2+\beta\left[(B+A)+(B-A)\alpha\right]} r \tag{13}$$

which, together yield (8).

Furthermore, it follows from Theorem 2 that

$$\sum_{n=1}^{\infty} n|a_n| \leq \frac{(B-A)\beta(1-\alpha)}{2+\beta\left[(B+A)+(B-A)\alpha\right]}. \tag{14}$$

Hence

$$|f'(z)| \leq \frac{1}{r^2} + \sum_{n=1}^{\infty} n|a_n| r^{n-1} \leq \frac{1}{r^2} + \sum_{n=1}^{\infty} n|a_n| \leq \frac{1}{r^2} + \frac{(B-A)\beta(1-\alpha)}{2+\beta\left[(B+A)+(B-A)\alpha\right]} \tag{15}$$

and

$$|f'(z)| \geq \frac{1}{r^2} - \sum_{n=1}^{\infty} n|a_n| r^{n-1} \geq \frac{1}{r^2} - \sum_{n=1}^{\infty} n|a_n| \geq \frac{1}{r^2} - \frac{(B-A)\beta(1-\alpha)}{2+\beta\left[(B+A)+(B-A)\alpha\right]} \tag{16}$$

which, together, yield (10). It can easily be seen that the function $f_1(z)$ defined by (9) is extremal for Theorem 3.

Corollary 2. *Let the function* $f(z)$ *defined by* (3) *be in the class* $T^*(A, B)$. *Then, for* $0 < |z| = r < 1$

$$\frac{1}{r} - \frac{B-A}{2+B+A} r \leq |f(z)| \leq \frac{1}{r} + \frac{B-A}{2+B+A} r$$

and

$$\frac{1}{r^2} - \frac{B-A}{2+B+A} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{B-A}{2+B+A}.$$

The result is sharp.

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БЕЛЕЖКИ ВЪРХУ ПОДКЛАС ОТ МЕРОМОРФНИ ФУНКЦИИ С ОТРИЦАТЕЛНИ КОЕФИЦИЕНТИ

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Предмет на тази статия е получаването на някои резултати относно коефициентите и свойствата на един клас мероморфни функции с отрицателни коефициенти.