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IS THE MORLEY'S THEOREM TRUE IN THE HYPERBOLIC GEOMETRY?*

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We prove that the Morley's threesector theorem which holds good in the Euclidean plane, is not true in the hyperbolic plane as well not true in the elliptic plane (geometry on the sphere). The validity of the Morley's theorem in the absolute geometry is equivalent to the Euclid's parallel postulate. We establish the sides of the Morley's threesector triangle are bounded from above with the constant arcch $\frac{17}{16}$.

1. Introduction. The great algebraic geometer Frank Morley (1860-1937) gave in 1900 the following brilliant assertion, which is now simply known as

Morley's threesector theorem. The three intersections of the threesectors of the angles of a triangle, lying near the three sides respectively, form an equilateral triangle [1].

G. Stanilov stated the problem: is this theorem true in the hyperbolic geometry?

2. Expressions of the sides of Morley's triangle. Let ΔABC is an arbitrary triangle in the hyperbolic plane with sides $AB = c, BC = a, CA = b$ and angles $\alpha = \sphericalangle A, \beta = \sphericalangle B, \gamma = \sphericalangle C$. Let AC_1 and AB_1 are the threesectors of the angle α, BC_1, BA_1

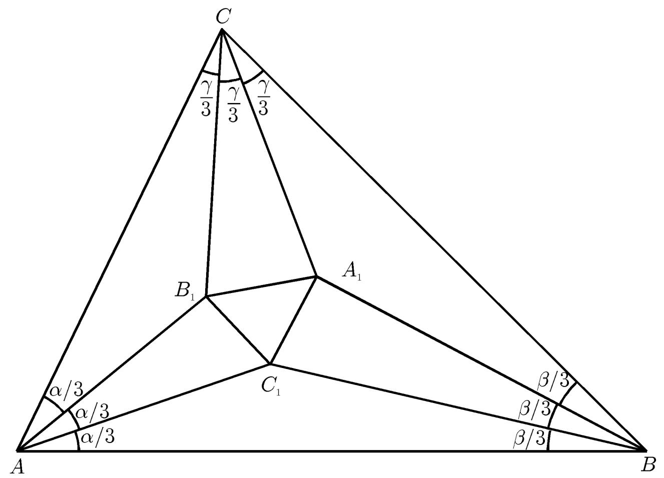

Fig. 1

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– of the angle β, and CA_1, CB_1 – of the angle γ. The $A_1B_1C_1$ is the Morley's triangle of the given triangle ABC (Figure 1).

At first we use the formula [2]

$$\operatorname{cotg}\alpha\sin\gamma = \operatorname{cth}\frac{a}{k}\operatorname{sh}\frac{b}{k} - \operatorname{ch}\frac{b}{k}\cos\frac{\alpha}{3}. \tag{1}$$

We apply this theorem two times for ΔABC_1:

$$\operatorname{cotg}\frac{\beta}{3}\sin\frac{\alpha}{3} = \operatorname{cth}\frac{AC_1}{k}\operatorname{sh}\frac{c}{k} - \operatorname{ch}\frac{c}{k}\cos\frac{\alpha}{3}, \tag{2}$$

$$\operatorname{cotg}\frac{\alpha}{3}\sin\frac{\beta}{3} = \operatorname{cth}\frac{BC_1}{k}\operatorname{sh}\frac{c}{k} - \operatorname{ch}\frac{c}{k}\cos\frac{\beta}{3}. \tag{3}$$

From (2) we find

$$\operatorname{ch}\frac{AC_1}{k} = \frac{\operatorname{ch}\frac{c}{k}\cos\frac{\alpha}{3}\sin\frac{\beta}{3} + \sin\frac{\alpha}{3}\cos\frac{\beta}{3}}{\operatorname{sh}\frac{c}{k}\sin\frac{\beta}{3}}\operatorname{sh}\frac{AC_1}{k} \tag{4}$$

and since

$$\operatorname{ch}^2\frac{AC_1}{k} - \operatorname{sh}^2\frac{AC_1}{k} = 1 \quad ,$$

we find

$$\operatorname{ch}\frac{AC_1}{k} = \frac{\operatorname{ch}\frac{c}{k}\cos\frac{\alpha}{3}\sin\frac{\beta}{3} + \sin\frac{\alpha}{3}\cos\frac{\beta}{3}}{\sqrt{\left(\operatorname{ch}\frac{c}{k}\cos\frac{\alpha}{3}\sin\frac{\beta}{3} + \sin\frac{\alpha}{3}\cos\frac{\beta}{3}\right)^2 - \operatorname{sh}^2\frac{c}{k}\sin^2\frac{\beta}{3}}}. \tag{5}$$

In the same way from ΔACB_1 we find

$$\operatorname{ch}\frac{AB_1}{k} = \frac{\operatorname{ch}\frac{b}{k}\cos\frac{\alpha}{3}\sin\frac{\gamma}{3} + \sin\frac{\alpha}{3}\cos\frac{\gamma}{3}}{\sqrt{\left(\operatorname{ch}\frac{b}{k}\cos\frac{\alpha}{3}\sin\frac{\gamma}{3} + \sin\frac{\alpha}{3}\cos\frac{\gamma}{3}\right)^2 - \operatorname{sh}^2\frac{b}{k}\sin^2\frac{\gamma}{3}}}. \tag{6}$$

From the cosine theorem for ΔAB_1C_1 we find

$$\operatorname{ch}\frac{B_1C_1}{k} = \operatorname{ch}\frac{AB_1}{k}\operatorname{ch}\frac{AC_1}{k} - \operatorname{sh}\frac{AB_1}{k}\operatorname{sh}\frac{AC_1}{k}\cos\frac{\alpha}{3} \tag{7}$$

and using the expressions (5) and (6) we get

$$\operatorname{ch}\frac{B_1C_1}{k} = \frac{P}{Q}, \tag{8}$$

where

$$\begin{aligned} P &= \left(\operatorname{ch}\frac{b}{k}\cos\frac{\alpha}{3}\sin\frac{\gamma}{3} + \sin\frac{\alpha}{3}\cos\frac{\gamma}{3}\right)\left(\operatorname{ch}\frac{c}{k}\cos\frac{\alpha}{3}\sin\frac{\beta}{3} + \sin\frac{\alpha}{3}\cos\frac{\beta}{3}\right) \\ &- \operatorname{sh}\frac{b}{k}\operatorname{sh}\frac{c}{k}\cos\frac{\alpha}{3}\sin\frac{\beta}{3}\sin\frac{\gamma}{3}, \end{aligned}$$

$$Q = \left\{\left[\left(\operatorname{ch}\frac{b}{k}\cos\frac{\alpha}{3}\sin\frac{\gamma}{3} + \sin\frac{\alpha}{3}\cos\frac{\gamma}{3}\right)^2 - \operatorname{sh}^2\frac{b}{k}\sin^2\frac{\gamma}{3}\right]\times\right.$$

$$\times \left[\left(\operatorname{ch} \frac{c}{k} \cos \frac{\alpha}{3} \sin \frac{\beta}{3} + \sin \frac{\alpha}{3} \cos \frac{\beta}{3} \right)^2 - \operatorname{sh}^2 \frac{c}{k} \sin^2 \frac{\beta}{3} \right] \Bigg\}^{1/2}$$

By symmetry we have the corresponding expressions for the $\operatorname{ch} \frac{A_1B_1}{k}$ and $\operatorname{ch} \frac{A_1C_1}{k}$.

3. Non-validity of Morley's theorem in the hyperbolic plane. After very long calculations we have proved that the equality

$$A_1B_1 = B_1C_1 \tag{9}$$

is equivalent to the equality

$$M \sin^2 \frac{\alpha}{3} \sin^4 \frac{\beta}{3} \sin^2 \frac{\gamma}{3} \sin \frac{\alpha+\gamma}{3} \sin \frac{\alpha-\gamma}{3} \xi\left(\alpha,\beta,\gamma\right) = 0, \tag{10}$$

where

$$M = \sin \frac{\delta}{2} \sin \left(\alpha + \frac{\delta}{2} \right) \sin \left(\beta + \frac{\delta}{2} \right) \sin \left(\gamma + \frac{\delta}{2} \right) \tag{11}$$

and $\xi\ (\alpha,\beta,\gamma)$ is non-identical zero function of α,β,γ. Then we can conclude:

I. If the triangle ABC is an isosceles triangle, then the same is true for $\Delta A_1B_1C_1$.

II. If the triangle ABC is an equilateral triangle, the same is true for $\Delta A_1B_1C_1$.

III. If (9) holds and $\Delta A_1B_1C_1$ is not an isosceles triangle ($\alpha \neq \gamma$), then it follows that the defect of the triangle ABC is zero.

In this way we give a new proof for the Morley's theorem in the Euclidean plane starting from the absolute geometry.

Theorem 1. *The Morley's threesector theorem is not true in the hyperbolic geometry.*

Theorem 2. *The validity of the Morley's theorem in the absolute geometry is equivalent to the Euclid's parallel postulate.*

Using the analogy between the formulas in the hyperbolic and in the elliptic trigonometry [2], we can formulate also

Theorem 3. *The Morley's theorem is not true also in the elliptic geometry.*

4. Above limit for the sides of the Morley's triangles. Using the expression (8) R. Hoefer at first gave some contra examples for the validity of Morley's theorem in the hyperbolic plane. He remarked that in many numerical cases the sides of the Morley's triangle are smaller from the number 0,35... . Then he did the following

Conjecture. The sides of the Morley's triangle for any triangle ABC in the hyperbolic plane are limited above from the constant $\operatorname{arcch} \frac{17}{16}$ (in the case when the constant $k = 1$).

We can prove this assertion for any equilateral triangle ABC. If $\alpha = \beta = \gamma = t$ we find

$$\operatorname{ch} B_1C_1 = \frac{\left(1 - 2\cos t + \cos \frac{t}{3}\right) \cos \frac{t}{3}}{\left(1 - 2\cos t + \cos^2 \frac{t}{3}\right)}, \tag{12}$$

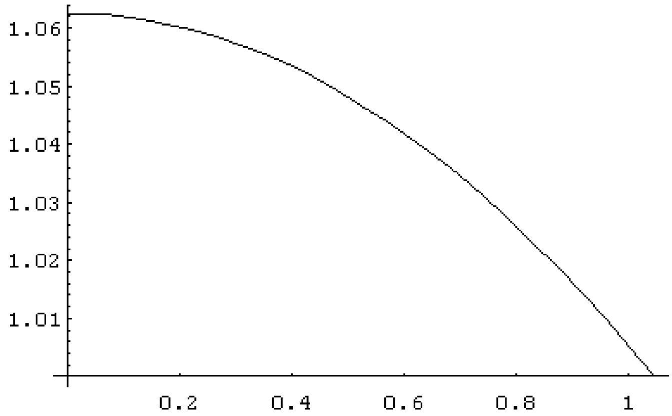


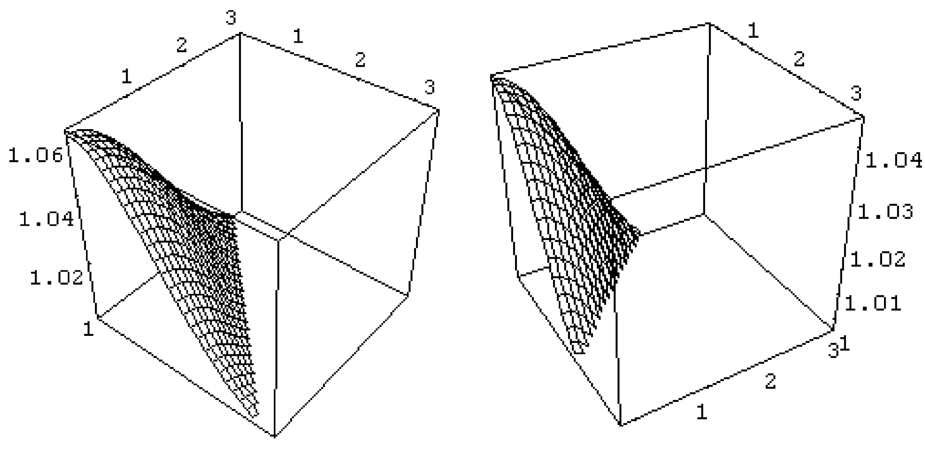
Fig. 2

which is equivalent to the equality

$$\operatorname{ch} B_1C_1 = \frac{4 + 7\cos\dfrac{t}{3} + 4\cos\dfrac{2t}{3} + 2\cos t}{5 + 7\cos\dfrac{t}{3} + 4\cos\dfrac{2t}{3}}. \tag{13}$$

Here t∈(0, $\pi/3$). The first derivative of this function is

$$(\operatorname{ch} B_1C_1)' = \frac{-\left(27\sin\dfrac{t}{3} + 36\sin\dfrac{2t}{3} + 30\sin t + 14\sin\dfrac{4t}{3} + 4\sin\dfrac{5t}{3}\right)}{3\left(5 + 7\cos\dfrac{t}{3} + 4\cos\dfrac{2t}{3}\right)^2}, \tag{14}$$



$\gamma = \pi/30$, $\alpha, \beta \in (0, \pi)$, $\alpha + \beta < \pi - \pi/30$ $\qquad$ $\gamma = \pi/3$, $\alpha, \beta \in (0, \pi)$, $\alpha + \beta < \pi - \pi/3$

Fig. 3

which means that the function is a decreasing function with

$$\lim_{\alpha\to 0} \operatorname{ch} B_1C_1 = \frac{17}{16}, \lim_{\alpha\to\frac{\pi}{3}} \operatorname{ch} B_1C_1 = 1. \tag{15}$$

Thus the conjecture for any equilateral triangle is proved. The graphic of the function (13) is given by figure 2.

For an arbitrary triangle ABC by computer graphic for many numerical cases for the angle γ and arbitrary α, β in the interval $(0, \pi)$ and $\alpha + \beta < \pi - \gamma$, we establish that $\operatorname{ch} B_1C_1$ is smaller than 1,062… . On Figure 3 are given two examples.

It shows the conjecture of R. Hoefer is appropriated.

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ВЯРНА ЛИ Е ТЕОРЕМАТА НА МОРЛИ В ХИПЕРБОЛИЧНАТА ГЕОМЕТРИЯ?

Грозьо Станилов, Десислава Йорданова, Роланд Хьфер, Юлиан Цанков

Доказваме, че теоремата на Морли, отнасяща се за трисектрисите на триъгълник в Евклидовата равнина, не е вярна в равнината на Лобачевски. Тя не е вярна и в елиптичната равнина (геометрията върху сферата). Валидността на теоремата на Морли в абсолютната геометрия е равносилна на Петия постулат на Евклид. Установяваме, че страните на триъгълниците на Морли са ограничени отгоре от константата $\operatorname{arcch} \frac{17}{16}$.