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PROBLEMS AND SOLUTIONS BY THE APPLICATION OF JULIA SET THEORY TO ONESTAGE AND MULTISTAGE NUMERICAL METHODS FOR SOLVING EQUATIONS

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In 1977 J. H. Hubbard developed the ideas of A. Cayley (1879) and solved the Newton–Fourier imaginary problem particular. We solve the Newton–Fourier and the Chebisheff–Fourier imaginary problems completely. It is known that the application of Julia set theory is possible to the onestage numerical method like the Newton's method for computing solution of the nonlinear equations. The secants method is twostage numerical method and the application of Julia set theory to it isn't demonstrated. Previously we have defined two onestage combinations: the Newton's–secants and the Chebisheff's–secants methods and have used the escape time algorithm to analyze the application of Julia set theory to these two combinations in some special cases. We consider and solve the Newton's–secants and Tchebicheff's–secants imaginary problems completely.

1. Introduction. In 1879 A. Cayley [1] demonstrated the Newton–Fourier imaginary problem for $F(z) = z^2 - C$. In 1977 J. H. Hubbard [3] solved this problem. Using the transformation $G(z) = \dfrac{z + C^{\frac{1}{2}}}{z - C^{\frac{1}{2}}} = u$, $G^{-1}(u) = C^{\frac{1}{2}}\dfrac{u+1}{u-1}$ he proved [1], [3] that the dynamical system

$$C : f_N(z) = z - \frac{F(z)}{F'(z)} \tag{1.1}$$

and $C :\ R(u) = u^2$ are equivalent: $R(u) = G \circ f_N \circ G^{-1}(u) = u^2$.

In [5] we define the Newton's–secants method for computing solutions of nonlinear equation $F(z) = 0$: it tell us to consider the dynamical system, associated with $F(z)$:

$$C : f_{Ns}(z) = f_N(z) - \frac{F(f_N(z))(f_N(z) - z)}{F(f_N(z)) - F(z)} \tag{1.2}$$

where $f_N(z)$ is the Newton transformation, associated with the function $F(z)$ (1.1).

In [6] we define the Chebisheff's secants method for computing solution of nonlinear equations: it is to consider the dynamic system:

$$C : f_{Chs}(z) = f_{Ch}(z) - \frac{F(f_{Ch}(z))(f_{Ch}(z) - z)}{F(f_{Ch}(z)) - F(z)},$$

where

$$f_{Ch}(z) = z - \frac{F(z)}{F'(z)} - \frac{F^2(z)F''(z)}{2(F'(z))^3} \tag{1.3}$$

2. Main results. Consider the polynomial

$$F(z) = z^n - C, \; n \in N \tag{2.1}$$

and the transformations:

$$G(z) = \frac{z - C^{\frac{1}{n}}}{z + C^{\frac{1}{n}}}, \; G^{-1}(u) = C^{\frac{1}{n}} \frac{u+1}{1-u}.$$

We can prove the following theorems.

Theorem 2.1. *The dynamical system* $f_N(z)$ (1.1) *where* $F(z)$ *is* (2.1) *and*

$$R_N(u) = G \circ f_N \circ G^{-1}(u) = \frac{2nu(u+1)^{n-1} - (u+1)^n + (1-u)^n}{2n(u+1)^{n-1} - (u+1)^n + (1-u)^n}$$

are locally equivalent in sufficiently little circle around each of the n roots of the polynomial $F(z) = z^n - C$*. Since* C *is generally a complex number the value* $C^{\frac{1}{n}}$ *is fixed as any of the* n *- different values of the* n *- roots of the polynomial* (2.1)*. The fixed points of* $R_n(u)$ *are* $u = 1$ *and* $u = -i \operatorname{tg} \dfrac{kpi}{n}$, $k = 1, 2, \ldots, n-1$

Theorem 2.2. *The dynamical system* (1.3) *where* $F(z)$ *is* (2.1) *and*

$$R_{Ch}(u) = \frac{(1-u^2)^n(4n-2) + (u-1)^{2n}(1-n) + (u+1)^{2n-1}\big(4n^2u - 3n(u+1) + u + 1\big)}{(1-u^2)^n(4n-2) + (u-1)^{2n}(1-n) + (u+1)^{2n-1}\Big(4n^2 - 3n(u+1) + u + 1\Big)}$$

are locally equivalent in sufficiently little open circle around each of the n roots of the polynomial (2.1)*. The fixed points of* (2.1) *are:*

$$1, \; -i \operatorname{tg} \frac{kpi}{n}, \; \frac{e^{\frac{i2pim}{n}} \dfrac{(n-1)^{\frac{1}{n}}}{(3n-1)^{\frac{1}{n}}} - 1}{e^{\frac{i2pim}{n}} \dfrac{(n-1)^{\frac{1}{n}}}{(3n-1)^{\frac{1}{n}}} + 1}, \; k = 1, 2, \ldots, n-1, \; m = 0, 1, 2, \ldots, n-1.$$

We can prove two similar theorems about $f_{Ns}(z)$ and $f_{Chs}(z)$ in the case (2.1), but the general formulas for $R_{Ns}(u)$ and $R_{Ch}(u)$ are too long and we will publish them later. Here we will consider only some examples. Consider now the polynomial:

$$F(z) = (z^n - C)^l, \; l, n \in N \tag{2.2}$$

In the well-known modifications of Newton's and Chebisheff's methods for solving the equation $F(z) = 0$ the following dynamic systems are considered:

$$C : f_{Nm}(z) = z - \frac{u(z)}{u'(z)}$$

and

$$C : f_{Chm}(z) = z - \frac{u(z)}{u'(z)} - \frac{u^2(z)u''(z)}{2u'(z)^3}$$

where $u(z) = \dfrac{F(z)}{F'(z)}$. We can prove two similar theorems about $f_{Nm}(z)$ and $f_{Chm}(z)$ in the case (2.2). We obtain that $R_{Nm}(z) = R_N(z)$, but the general formula for $R_{Chm}(z)$ is too long and we will publish it later. We will consider here only some examples.

3. Examples and the application of Julia set theory.

1) $n = 2$:

a) Newton's–secants method:

$$f_{Ns}(z) = \frac{z^3 + 3Cz}{3z^2 + C}$$

The fixed points of $f_{Ns}(z)$ are 0, $C^{\frac{1}{2}}$ and $-C^{\frac{1}{2}}$. The first is repulsive: $f'_{Ns}(0) = 3$ and the other are attractive:$f'_{Ns}(C^{\frac{1}{2}}) = f'_{Ns}(-C^{\frac{1}{2}}) = f''_{Ns}(C^{\frac{1}{2}}) = f''_{Ns}(-C^{\frac{1}{2}}) = 0$, but $f'''_{Ns}(C^{\frac{1}{2}}) = f'''_{Ns}(-C^{\frac{1}{2}}) = \dfrac{3}{2C}$. This is clear, because $R_{Ns}(u) = u^3$. The Julia set for u^3 is $|u| = 1$. This is the second case in the fractal geometry that the formula for Julia set is the same and too simple (see [1], [2] and [3]).

It is clear too that the order of successive approximations is 3 [4], greater than the order of Newton's approximations which is 2 and the order of secants' approximations which is 1,61803... [4]. Let's consider the problem about the computation's efficiency in this case [3]. Assume that for the computation of $F(z)$, $F'(z)$ and $F(f_n(z))$ are necessary 3 computation's units. Then the efficiency of the Newton's–secants method will be $3^{\frac{1}{3}} \approx 1,442...$ that is between the efficiency of the secants methods (1,61803...) and the sufficiently of the Newton's method ($2^{\frac{1}{2}} \approx 1,414...$). Assume that 4 computation units are used then the efficiency of the Newton–secants method will be $3^{\frac{1}{4}}$ that is more than the efficiency of the chords method which is one.

b) Chebisheff's–secants method:

$$R_{Chs}(u) = \frac{u^4(2 + u)}{1 + 2u}.$$

The fixed points of $R_{Chs}(u)$ are $0, 1$ and -1. The first is attractive and in a little circle of 0 $R_{Chs}(u)$ and $R(u) = u^3$ are equivalent. The order of successive approximations is 4, this is greater than the Chebisheff's approximations order which is 3 [4]. The point -1 is parabolic and the point 1 is repulsive. The fixed point's type is determined by the facts that $R'_{Chs}(0) = R''_{Chs}(0) = R'''_{Chs}(0) = 0$, $R^{(4)}_{Chs}(0) = 48$, $R_{Chs}(1) = \dfrac{11}{3}$, $R'_{Chs}(-1) = 1$.

On the plate 1 the Escape time algorithm [2] is used to analyze the trajectories of $R_{Chs}(u)$.

c) Chebisheff's–modification method:

$$R_{Chm}(z) = \frac{u^4(u^2 + 3)}{3u^2 + 1}$$

2) $n = 3$:

a) Newton's method:

$$R_N(u) = \frac{2u^2(u + 3)}{3 + 3u + 3u^2 - u^3}$$

On the plate 2 the escape time algorithm [2] is used to analyze the trajectories of $R_N(u)$.

b) Chebisheff's method:

$$R_{Ch}(u) = \frac{4u^3(u^3 + 9u^2 + 15u + 15)}{9 + 36u + 45u^2 + 60u3 + 15u^4 - 5u^6}$$

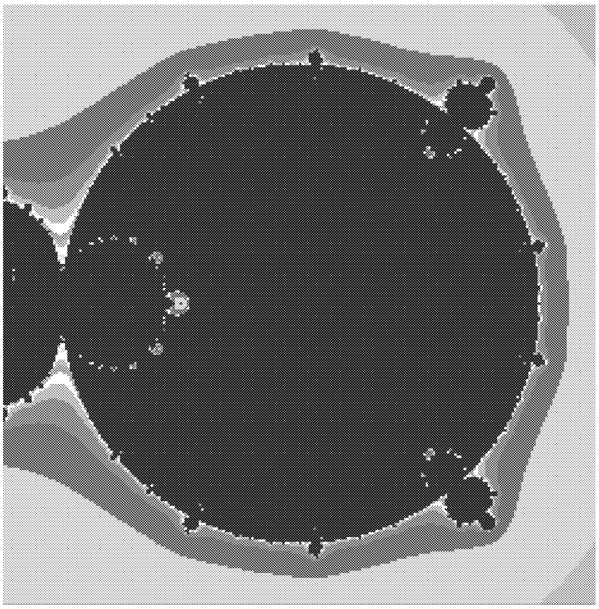

Plate 1

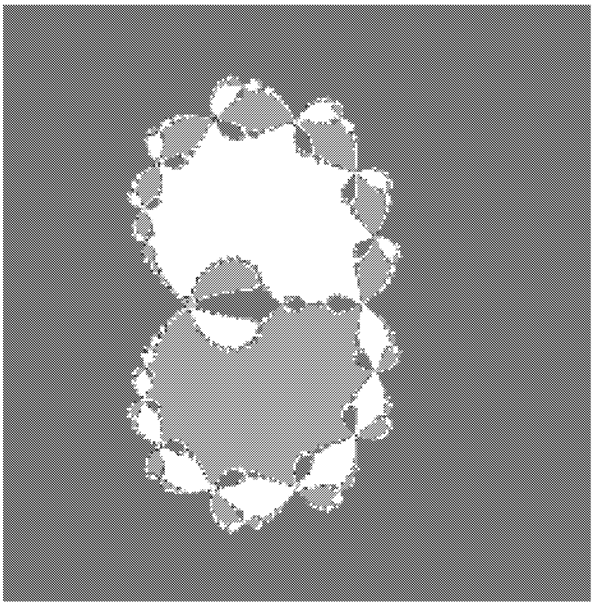

Plate 2

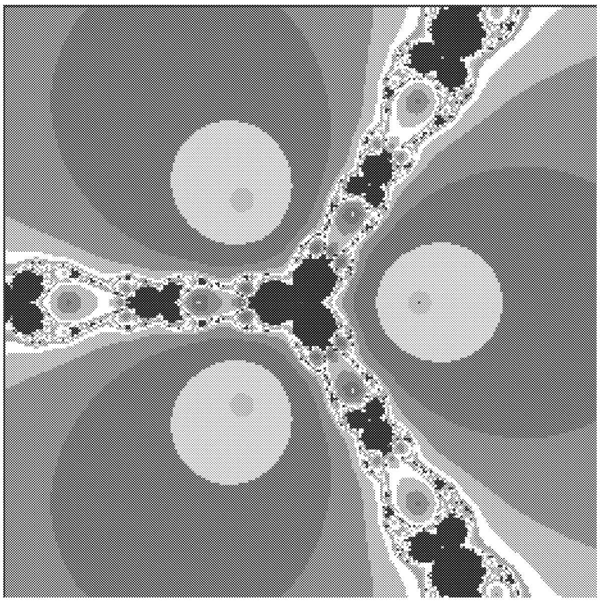

Plate 3

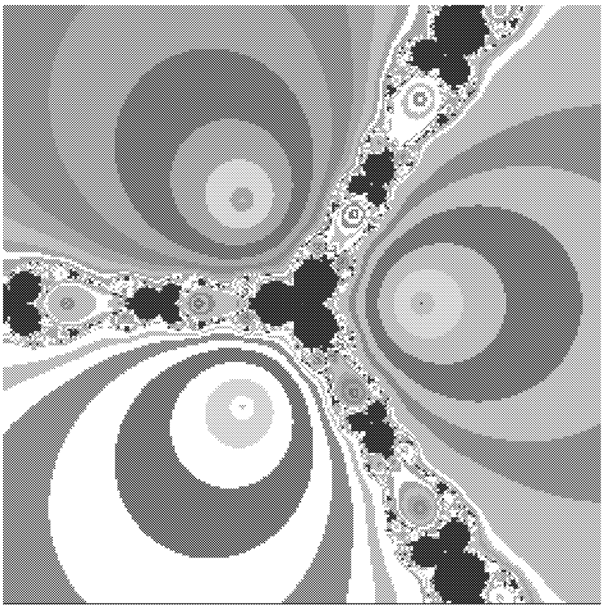

Plate 4

c) Newton's–secants method:

$$R_{Ns}(u) = \frac{4u^3(u^4 + 18u^3 + 60u^2 + 54u + 27)}{27 + 81u + 171u^2 + 189u^3 + 177u^4 + 27u^5 - 23u^6 - 9u^7}$$

On the plates 3, 4 the ESCAPE TIME ALGORITHM [2] is used to analyze the trajectories of $f_{Ns}(z)$ in the case: $F(z) = z^3 - 1$.

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ПРОБЛЕМИ И РЕШЕНИЯ ПРИ ПРИЛОЖЕНИЕТО НА ТЕОРИЯТА НА МНОЖЕСТВАТА НА ЖЮЛИА КЪМ ЕДНОТОЧКОВИТЕ ЧИСЛЕНИ МЕТОДИ ЗА РЕШАВАНЕ НА УРАВНЕНИЯ

Анна Вълкова Томова

В 1977 г. Дж. Х. Хабард развива идеите на А. Кели (1879) и решава имагинерния проблем на Нютон–Фурие частично. Ние решаваме имагинерния проблем Нютон–Фурие и Чебишев–Фурие в общия случай. Както е известно, приложението на теорията на множествата на Жюлиа е възможно към едноточкови числени методи, какъвто е методът на Нютон за решаване на нелинейни уравнения. Методът на секущите е двуточков и досега не е публикувано приложение на теорията на множествата на Жулиа към него. Предварително сме дефинирали две едноточкови комбинации: метод на Нютон–секущи и метод на Чебишев–секущи и сме използвали the ESCAPE TIME ALGORITHM, за да анализираме приложението на теорията на множествата на Жюлиа към тези две комбинации в някои частни случаи. Тук разглеждаме и решаване на имагинерния проблем на Нютон-секущи и Чебишев–секущи в общия случай.