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STANILOV MANIFOLDS AND THEIR CHARACTERIZATION IN DIMENSION FOUR*

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Let (M, g) be an n-dimensional Riemannian manifold, E^k be an arbitrary k-dimensional subspace of the tangent space M_p, $e_1, e_2, \dots, e_k$ be an arbitrary orthonormal basis in E^k and let $S(E^k)$ be a linear symmetric operator of M_p defined by $S(E^k)(u) = S(e_1, e_2, \dots, e_k)(u) = \Sigma_{i<j} R(e_i, e_j, R(e_i, e_j)u)$, $i, j = 1, 2, \dots, k$. We say that (M, g) is $k - S$ or k-Stanilov manifold if the eigenvalues of the curvature operator $S(E^k)$ are pointwise constants at any point $p \in M$. In the present note we proved that a four-dimensional Riemannian manifold (M, g) is $k - S$ manifold for $k = 2, 3$ if and only if (M, g) is a space of constant sectional curvature or (M, g) is locally isometric to a warped product of the form $B \times_f N$ where B is 1-dimensional subspace of M_p and N is 3-dimensional space form of constant sectional curvature K and warped smooth function on B is given by $f(t) = \sqrt{Kt^2 + Ct + D}$ where $K,\ C,\ D$ are constants such that $C^2 - 4KD \neq 0$. Thus we fully characterize four-dimensional $k - S$ manifolds.

Let (M, g) be a four-dimensional Riemannian manifold with metric g, curvature tensor R and let p be a point of M. The skew-symmetric curvature operator $R(E^2)$ of the tangent space M_p to M at a point $p \in M$ is a skew-symmetric linear mapping

$$R\left(E^2\right) : M_p \to M_p$$

defined by

$$R\left(E^2\right)(u) = R\left(X, Y, u\right),$$

where $E^2 = E^2(p; X, Y)$ is an arbitrary two-dimensional tangent plane of M_p. It is easy to see that this operator does not depend on the orthonormal oriented basis in the plane E^2. This curvature operator was defined from G. Stanilov which first state a problem for the investigation of a four-dimensional Riemannian manifolds of pointwise constant curvature eigenvalues of $R(E^2)$ and he proved in a joint work with R. Ivanova the following assertion [1]: The curvature operator $R(E^2)$ has pointwise constant eigenvalues at any point p of a four-dimensional Einstein Riemannian manifold (M, g) if and only if (M, g) is a space of constant sectional curvature.

Further manifolds where the skew-symmetric curvature operator $R(E^2)$ has pointwise constant eigenvalues were called from P.B. Gilkey as IP (Ivanov, Petrova) manifolds. We believe the main reason is the following result:

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Theorem A [2]. *Let (M,g) be a four-dimensional Riemannian manifold such that eigenvalues of the skew-symmetric curvature operator $R(E^2)$ are pointwise constants at any point p of the manifold. Then (M,g) is locally (almost everywhere) isometric to one of the following spaces:*

a) *real space form;*

b) *a warped product $B\times_f N$ where B is open interval on the real line, N is 3-dimensional space form of constant sectional curvature K and f is a smooth function on B given by $f(x)=\sqrt{Kx^2+Cx+D}$ where K,C,D are constant such that $C^2-4KD\neq 0$.*

From [2] we will use the following:

Proposition 2.3. *Let (M,g) be a four-dimensional Riemannian manifold such that the eigenvalues of the skew-symmetric curvature operator $R(E^2)$ are pointwise constants at any point p of the manifold. Then at any point $p\in M$ there exist an orthonormal basis e_1,e_2,e_3,e_4 in the tangent space M_p such that $R_{ijjk}=0$, $R_{ijks}=0$ and exactly one of the following cases occurs:*

$$\begin{aligned} &\text{i)} && K_{12}=K_{13}=K_{14}=K_{23}=K_{24}=K_{34},\\ &\text{ii)} && K_{12}=K_{13}=-K_{14}=-K_{23}=K_{24}=K_{34},\\ &\text{iii)} && K_{12}=K_{13}=K_{14}=-K_{23}=-K_{24}=-K_{34}.\end{aligned}$$

The classification of S. Ivanov and I. Petrova was extended by P. Gilkey, J. Leahy, U. Semmelman and H. Sadofsky [3], [4], [5]. We summarize these results as follows:

Theorem B. *Let (M,g) be a Riemannian manifold of dimension $m\geq 5$ and $m\neq 7$ such that (M,g) is IP manifold. Then either (M,g) has constant sectional curvature or (M,g) is locally isometric to a warped product of the form*

$$ds^2=dt^2+f(t)ds_N^2 \ \text{on}\ (t_0,t_1)\times N, \tag{1}$$

where $f(t)=\frac{1}{2}(Kt^2+At+B)>0$, and ds_N^2 has constant sectional curvature K.

In the present note we generalize IP-conjecture in Riemannian geometry using a curvature operator $S(E^k)$ defined from G. Stanilov in the following way:

Definition 1. *Let (M,g) be an n-dimensional Riemannian manifold, E^k be an arbitrary k-dimensional subspace of the tangent space M_p, $e_1,e_2,\dots,e_k$ be an arbitrary orthonormal basis in E^k and let $S(E^k)$ be the linear symmetric operator of the tangent space M_p defined by*

$$S\left(E^k\right)(u)=S\left(e_1,e_2,\dots,e_k\right)(u)=\sum_{\substack{i,j=1\\ i<j}}^{k} R\left(e_i,e_j,R\left(e_i,e_j,u\right)\right).$$

We say that (M,g) is $k-S$ manifold if the eigenvalues of the curvature operator $S(E^k)$ are pointwise constants at any point $p\in M$.

It is easy to check that $S(E^k)$ does not depend on the basis $e_1,e_2,\dots,e_k$ in E^k. Evidently in the case $\dim M=4$ it is possible to exist only $2-S$ and $3-S$ manifolds. Following theorems A and B we can prove that if (M,g) is $2-S$ (or IP) manifold, then (M,g) is $3-S$ manifold. Further we will prove that converse result is true.

Let e_1,e_2,e_3,e_4 be an orthonormal basis in the tangent space M_p at a point $p\in M$. If $x,\ y$ is orthonormal pair of vectors in the tangent space M_p, then the matrix of the curvature operator $S(x,y) = (A)$ with respect to this basis has the entries:

$$(2)\quad \begin{aligned}
a_{11} &= -R^2(x,y,e_1,e_2)^- R^2(x,y,e_1,e_3) - R^2(x,y,e_1,e_4)\\
a_{12} &= -R(x,y,e_1,e_3).R(x,y,e_2,e_3) - R(x,y,e_1,e_4).R(x,y,e_2,e_4)\\
a_{13} &= R(x,y,e_1,e_2).R(x,y,e_2,e_3) - R(x,y,e_1,e_4).R(x,y,e_3,e_4)\\
a_{14} &= R(x,y,e_1,e_2).R(x,y,e_2,e_4) + R(x,y,e_1,e_3).R(x,y,e_3,e_4)\\
a_{22} &= -R^2(x,y,e_1,e_2) - R^2(x,y,e_2,e_3) - R^2(x,y,e_2,e_4)\\
a_{23} &= -R(x,y,e_1,e_2).R(x,y,e_1,e_3) - R(x,y,e_2,e_4).R(x,y,e_3,e_4)\\
a_{24} &= -R(x,y,e_1,e_2).R(x,y,e_1,e_4) + R(x,y,e_2,e_3).R(x,y,e_3,e_4)\\
a_{33} &= -R^2(x,y,e_1,e_3)^- R^2(x,y,e_2,e_3) - R^2(x,y,e_3,e_4)\\
a_{34} &= -R(x,y,e_1,e_3).R(x,y,e_1,e_4) - R(x,y,e_2,e_3).R(x,y,e_2,e_4)\\
a_{44} &= -R^2(x,y,e_1,e_4)^- R^2(x,y,e_2,e_4) - R^2(x,y,e_3,e_4).
\end{aligned}$$

From our assumption (M,g) to be $3-S$ manifold we have the relations

$$\operatorname{trace} S(x,y,z) = \operatorname{trace} S(x,y,u), \quad \operatorname{trace} S(x,u,z) = \operatorname{trace} S(y,u,z),$$

where $x,\ y,\ z,\ u$ is an arbitrary orthonormal quadruple in M_p. These equalities give us the system

$$\begin{aligned}
\operatorname{trace} S\,(x,z) + \operatorname{trace} S\,(y,z) &= \operatorname{trace} S\,(x,u) + \operatorname{trace}\,(y,u)\,,\\
\operatorname{trace} S\,(x,u) + \operatorname{trace} S\,(x,z) &= \operatorname{trace} S\,(y,u) + \operatorname{trace}\,(y,z)\,,
\end{aligned}$$

from which it follows

$$(3)\qquad \operatorname{trace} S\,(x,y) = \operatorname{trace} S\,(z,u)\,.$$

From here using (2) and putting $x = e_1,\ y = e_2,\ z = e_3,\ u = e_4$ we obtain

$$(4)\quad \begin{aligned}
&K^2(x,y) + R^2(y,x,x,z) + R^2(y,x,x,u) + R^2(x,y,y,z) + R^2(x,y,y,u) =\\
&= K^2(z,u) + R^2(x,u,u,z) + R^2(y,z,z,u) + R^2(y,u,u,z) + R^2(x,z,z,u),
\end{aligned}$$

where K is the sectional curvature function on M. Changing in (4) x by $ax + bz$ and z by $-bx + az$, where a and b are arbitrary real numbers such that $a^2 + b^2 = 1$ and making some similarly calculations we obtain the relation:

$$(5)\qquad a^2A + b^2B + C = 0,$$

where

$$(6)\quad \begin{aligned}
A \;&=\; 4K(x,y)R(x,y,y,z) + 4R(y,x,x,u)(R(y,x,z,u) + R(y,z,x,u))\\
&\quad +2(K(y,z) - K(y,x))R(x,y,y,z) + 2(K(u,z) + K(u,x))R(x,u,u,z),\\
B \;&=\; 4K(z,y)R(z,y,y,x) + 4R(y,z,z,u)(R(y,z,x,u) + R(y,x,z,u))\\
&\quad +2(K(y,x) - K(y,z)).R(z,y,y,x) + 2(K(u,z) + K(u,x))R(x,u,u,z)\\
C \;&=\; R(x,y,y,u)R(z,y,y,u) - R(y,x,x,u)R(y,z,z,x)\\
&\quad -R(z,x,x,u)R(x,z,z,u) + R(y,z,z,u)R(y,u,u,x).
\end{aligned}$$

Because of $a^2 = 1 - b^2$, then from (5) we get $A = B = -C$. Now using (6) and some properties of the Riemannian curvature tensor R [7] we obtain the equation

$$(R(y,x,x,u) - R(y,z,z,u)).(R(y,x,z,u) + R(y,z,x,u)) = 0$$

and evidently at least one of the possibilities are satisfied:

$$(7)\qquad R(y,x,x,u) - R(y,z,z,u) = 0,$$

(8) $$R(y,x,z,u)+R(y,z,x,u)=0.$$

We can check that these relations are equivalents each other. Using first Bianchi identity and (8) we get

(9) $$R(x,z,y,u)=0.$$

This equality holds for any orthonormal quadruple of tangent vectors $x,\ y,\ z,\ u$ in the tangent space M_p, at any point $p\in M$. From the Schouten's theorem [8] follows that (M,g) is a conformally flat Riemannian manifold. Then for the curvature tensor R it holds the representation [6]

$$2.R(x,y,z,u)=g(y,z)\rho(x,u)-\rho(x,z)g(y,u)-g(x,z)\rho(y,u)+\rho(y,z)g(x,u).$$

For the curvature operator $\Re$ of the second exterior product $\wedge^2(M_p)$ on M we obtain

$$2\Re\left(x\wedge y\right)=\left(\rho\left(x\right)\wedge y-\rho\left(y\right)\wedge x\right)-\frac{\tau}{6}$$

where τ is the scalar curvature. From here it is easy to see that if e_1,e_2,e_3,e_4 are eigenvectors of the Ricci operator ρ with the corresponding eigenvalues $\lambda_1,\lambda_2,\lambda_3,\lambda_4$, then 2-vectors $e_1\wedge e_2$, $e_1\wedge e_3$, $e_1\wedge e_4$, $e_2\wedge e_3$, $e_2\wedge e_4$, $e_3\wedge e_4$ are eigenvectors of the curvature operator $\Re$ where $e_i\wedge e_j$ has the corresponding eigenvalue $a_{ij}=\frac{1}{2}\left(\lambda_i+\lambda_j-\frac{\tau}{3}\right)$, $i,j=1,2,3,4$. Using this fact either (4) and (9), for all the curvature components we have formulas

(10) $$K_{ij}^2=K_{st}^2$$

for all different i,j,s,k and $R_{ijst}=0$ – otherwise.

From (10) using that $a_{ij}=K_{ij}$, we get the relation $(\lambda_i+\lambda_j-\frac{\tau}{3})^2=(\lambda_s+\lambda_t-\frac{\tau}{3})^2$, hence $\tau(\lambda_i+\lambda_j-\lambda_s-\lambda_t)=0$. From here it follows two cases: $\lambda_i+\lambda_j-\lambda_s-\lambda_t=0$ or $\tau=0$. In the first case we have the system

$$\lambda_1+\lambda_2=\lambda_3+\lambda_4,\quad \lambda_1+\lambda_3=\lambda_2+\lambda_4,\quad \lambda_1+\lambda_4=\lambda_2+\lambda_3.$$

From this system it follows that $\lambda_1=\lambda_2=\lambda_3=\lambda_4$ and it is easy to see that (M,g) is a Riemannian manifold of constant sectional curvature.

In the second case when $\tau=0$ we have that the curvature tensor R belongs to the projections R_4 in the standard decomposition $R=R_1\oplus R_2\oplus R_3\oplus R_4$ of the curvature tensor R in four-dimensional Riemannian geometry [6]. It is well-known that $R\in R_4$ if and only if at any point $p\in M$ and for any tangent plane $\alpha\in M_p$ holds the equation

(11) $$\sigma\left(\alpha\right)=-\sigma\left(\alpha^{\perp}\right).$$

Having in mind (10) and (11) for the characteristic equation of the curvature operator $S(e_1,e_2,ae_3+be_4)$ (with root c) we obtain

$$\left(K_{12}^2+a^2K_{13}^2+b^2K_{23}^2+c\right).\left(K_{12}^2+b^2K_{13}^2+a^2K_{23}^2+c\right)\times$$
$$\times\left(c^2+\left(K_{13}^2+K_{23}^2\right)c+a^2b^2\left(K_{13}^2-K_{23}^2\right)\right)=0.$$

From our assumption (M,g) to be $3-S$ manifold it follows that the eigenvalues of the curvature $S(e_1,e_2,ae_3+be_4)$ are pointwise constants. Then using the last characteristic equation we can easy prove that $K_{12}^2=K_{13}^2$. Analogously we obtain $K_{12}^2=K_{23}^2$ and hence

(12) $$K_{12}^2=K_{13}^2=K_{23}^2.$$

It is clear that from (11) and (12) we have one of the equalities in Proposition 2.3

which means that (M, g) is $2-S$. Thus we prove our main result

Theorem C. *Let (M, g) be a four-dimensional Riemannian manifold. Then (M, g) is $3-S$ manifold if and only if (M, g) is $2-S$ manifold.*

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МНОГООБРАЗИЯ НА СТАНИЛОВ И ХАРАКТЕРИЗИРАНЕТО ИМ В ЧЕТИРИМЕРНИЯ СЛУЧАЙ

Веселин Видев, Юлиан Цанков

Нека (M, g) е n-мерно Риманово многообразие, E^k е k-мерно подпространство на допирателното пространство M_p, $e_1, e_2, \ldots, e_k$ е ортонормирана база в E^k и $S(E^k)$ е линеен симетричен оператор в M_p, дефиниран чрез $S(E^k)(u) = \sum_{i<j} R(e_i, e_j, R(e_i, e_j, u))$, $i, j = 1, 2, \ldots, k$. Казваме, че (M, g) е $k-S$ или k-Станилово многообразие, ако собствените стойности на кривинния оператор $S(E^k)$ са точково постоянни във всяка точка от M. В настоящата работа е доказано, че едно 4-мерно Риманово многообразие е $k-S$ многообразие за $k = 2, 3$ тогава и само тогава, когато то е пространство с постоянна секционна кривина или е локално изометрично на изкривеното произведение, описано в Теорема В.