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DIOPHANTIAN FIGURES AND DIOPHANTIAN CARPETS

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In this paper we expose different constructions of Diophantian figures obtained with the help of computer experiments and geometric considerations. The notion of Diophantian figure was introduced in [1] and developed in [2].
AMS Subject Classification: 05C12, 11A99, 05B99

1. Recall of definitions. We shall consider the so-called Diophantian plane, i.e. the Cartesian product $\mathbb{Z} \times \mathbb{Z}$, where by $\mathbb{Z}$ is denoted the ring of integers. Clearly, Diophantian plane is the lattice of all points in the plane of Decartes $\mathbb{R} \times \mathbb{R}$, ($\mathbb{R}$ is the field of real numbers) with integer coordinates (x, y), $x \in \mathbb{Z}$, $y \in \mathbb{Z}$.

We recall that *Diophantian figure* is defined by a set of points in Diophantian plane under the condition that the distance between each couple of its points is a positive integer. A Diophantian figure is called linear if its points lie on a line in the plane of Decartes. In the contrary, i.e. in the case the figure contains at least three different non-collinear points, we say that we have a flat Diophantian figure.

According to a theorem of P. Erdös each Diophantian figure defined by an infinite number of different points is linear. So, flat Diophantian figures are always with a finite number of points.

2. Diophantian triangles. Diophantian figures admit triangulation with Diophantian triangles [2], called Diophantian triangulation. This implies that these figures can be constructed by Diophantian triangles.

A large class of Diophantian figures can be obtained from Pythagorean triangles with common cathetus (leg).

Proposition 1. *The set of all Pythagorean triangles with fixed common cathetus is finite.*

Proof. Let $(\mathbf{a}, y, z)$ be an arbitrary Pythagorean triple with fixed cathetus $\mathbf{a}$. Here by z is denoted the hypotenuse and by y the other cathetus.

We shall consider the equality

$$\mathbf{a}^2 = z^2 - y^2 = (z - y)(z + y).$$

Clearly $z - y$ and $z + y$ must take a finite numbers of integer values, as they are divisors of $\mathbf{a}^2$. Indeed, if $z - y = \mathbf{p}_k$, and $z + y = \mathbf{q}_l$, $\mathbf{a}^2 = \mathbf{p}_k\,\mathbf{q}_l$ we obtain

$$\mathbf{z} = 1/2(\mathbf{p}_k + \mathbf{q}_l) \qquad \text{and} \qquad \mathbf{y} = 1/2(\mathbf{p}_k - \mathbf{q}_l),$$

where k and l are indices which take a finite numbers of values. □

Remark. Proposition 1 follows and by Erdös theorem cited above.

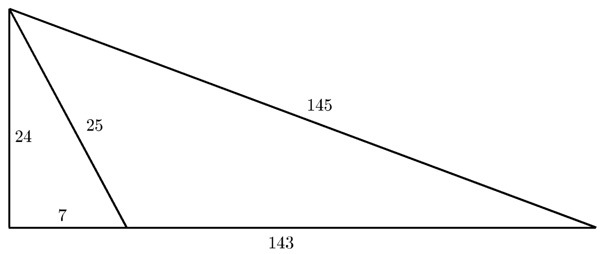
Corollary. Each system of Pythagorean triangles with common cathetus determines a Diophantian figure. The examples below are obtained by computer program selecting Pythagorean triples with common cathetus.

(**24**, 143, 145), (**24**, 7, 25),

(**660**, 12091, 121090), (**660**, 4331, 4381), (**660**, 2989, 3061),
(**660**, 989, 1189), (**660**, 779, 1021), (**660**, 259, 709).

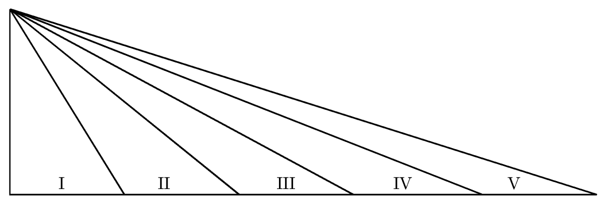
(**840**, 19591, 19609), (**840**, 11009, 11041), (**840**, 7031, 7081),
(**840**, 3551, 3649), (**840**, 1081, 1369), (**840**, 559, 1009),
(**840**, 41, 841).

We give a sketch for the first example. It defines a Diophantian figure with 4 points (vertices)



In the case of second example, a Diophantian figure with 7 vertices is defined, in the third case – with 8 vertices.

In all cases a triangulation composed by one Pythagorean triangle and other Diophantian ones (respectively 1, 6 and 7) is determined.



A mathematical algorithm can be developed as follows. Setting

$$\mathbf{a} = a_1^{\alpha_1} a_2^{\alpha_2} \dots a_n^{\alpha_n}, \text{ i.e.} \mathbf{a}^2 = a_1^{2\alpha_1} a_2^{2\alpha_2} \dots a_n^{2\alpha_n}$$

we can compare the divisors of $\mathbf{a}^2$ with the divisors of the product $(z-y)(z+y)$.

As a result we obtain a number of equations of the following form

$$z - y = \dots \quad \text{and} \quad z + y = \dots$$

Solving these equations we obtain the result.

The number of divisors of $\mathbf{a}^2$ is equal to $(2\alpha_1+1)(2\alpha_2+1)\dots(2\alpha_n+1)$, which shows that the number of possible systems for z and y is very big, when α_k are large enough. However we take the following example: $\mathbf{a} = 24$, $\mathbf{a}^2 = 576 = 2^6 3^2$. Here we have

$(2.3+1)(2+1)=21$ divisors, and respectively a list of system for z and y as above. For instance

$z-y=1$, $z+y=2^6 3^2$ which gives non-integer solutions, and many other systems which do not give integer solutions, but we have

$z-y=2.3^2$, $z+y=2^5$ which gives $z=25$, $y=7$,

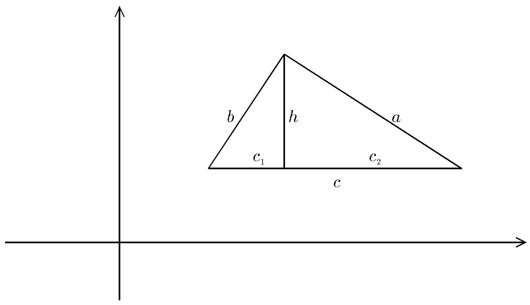
$z-y=2$, $z+y=2^5 3^2$ which give $z=145$, $y=143$. □

Proposition 2. *If (a,b,c) are lengths of the sides of a Diophantian triangle, then $a^2+b^2+c^2$ is an even number.*

Proof. In the case of Pythagorean triangle we have

$$a^2+b^2+c^2=2c^2.$$

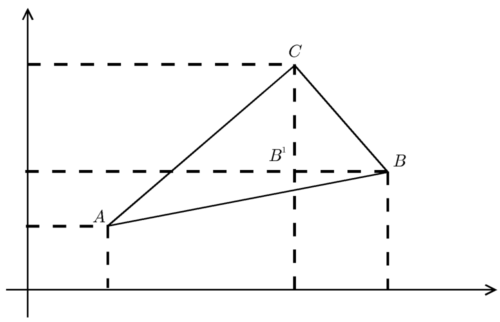
In the case of a horizontal side



we have

$$a^2+b^2+c^2=2h^2+2c^2-2c_1c_2.$$

In the general case of Diophatian triangle ABC with coordinates of the vertices $A(a_1,a_2)$, $B(b_1,b_2)$, $C(c_1,c_2)$



we have

$$a^2+b^2+c^2=2(a_1^2+a_2^2+b_1^2+b_2^2+c_1^2+c_2^2)-2(a_1b_1+a_1c_1+b_1c_1)-2(a_2b_2+a_2c_2+b_2c_2).$$

□

Corollary. In each Diophantian triangle there are always an even number of sides with odd length, i.e. 0 or 2. The distribution of the parity for the sides in a Diophantian figure with fixed Diophantian triangulation can be reconstructed starting by an arbitrary component of the considered triangulation.

3. Big Pythagorean triangles: a hypothesis. There are Pythagorean triangles with an arbitrary large length of its sides.

Now we introduce the following integer-valued function $k = \pi(n)$, $n \in \mathbb{N}$.

By definition

$\pi(n) = 0$ if n is not a cathetus of Pythagorean triangle,
$\pi(n)$ is the number of all Pythagorean triangles with n as a cathetus.

According some computer consideration on Pythagorean triples, here we state the following hypothesis: the function $k = \pi(n)$ is a slowly increasing function and it is true that

$$\lim_{n\to\infty} (\pi(n)/n) = 0.$$

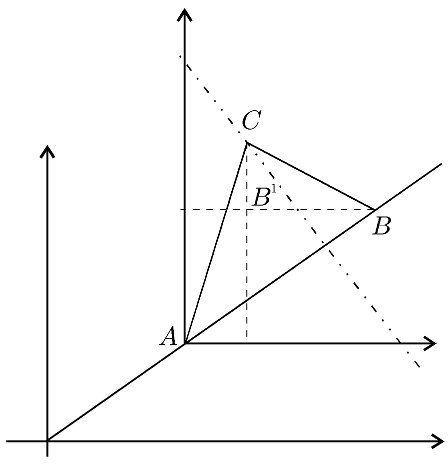
4. A Diophantian equation. Having in mind the condition of the Proposition 2, here we can accept that $a_1 = a_2 = 0$.

Proposition 4. *Setting $c_1 = x_1$, $c_2 = x_2$, we obtain the following Diophantian equation of first degree for x_1 and x_2*

$$2b_1x_1 + 2b_2x_2 = c^2 + b^2 - a^2.$$

For given coordinates (b_1, b_2) (i.e. for given position and length c of the segment AB) we have that the solutions of the obtained Diophantian equations, if they exist, determine points on a line, which is perpendicular to AB, the lengths a and b being arbitrary fixed.

Proof. Clearly, we have $c^2 = b_1^2 + b_2^2$ and $b_2 = x_1^2 + x_2^2$. On the other hand we have $a^2 = (b_1 - x_1)^2 + (b_2 - x_2)^2$ (See the triangle B^1BC on the draw below). The above mentioned three equalities implies the proposition.



According to the general theory of Diophantian equations of first degree, if (x_1^0, x_2^0) is one solution, then all solutions are given by the following formulas [3]

$$x_1 = x_1^0 + b_2/\langle b_1, b_2\rangle t,\ x_2 = x_2^0 - b1/\langle b_1, b_2\rangle t,$$

where t is an integer, and $\langle b_1, b_2\rangle$ is the greatest common divisor of b_1 and b_2.

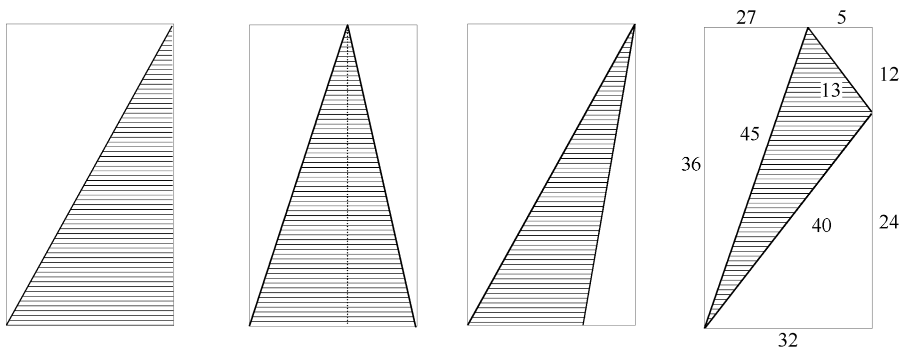
Eliminating t from the above two equations we receive for the solutions (x_1, x_2) that

$$x_2 - x_2^0 = -b_1/b_2(x_1 - x_1^0).$$

This is an equation of a line, which passes through the point (x_1^0, x_2^0), perpendicularly to the line of the segment AB. $\square$

Remark. This proposition concerns the construction of a Diophantian triangle with given sides a, b, c and given coordinates (b_1, b_2). The analogous construction in the classical planimetry is well known. Our construction don't follows from the classical one. It is of arithmetical character. Of course, the number of solutions is not infinite, when c, a, b are fixed. The solutions are less then two. It was shown that the line determined by these two points is perpendicular to the side AB as in the classical case. The above written Diophantian equation is not always solvable. For instance, for its solvability it is necessary the lengths b and c to be of the same parity. Indeed, $c^2 + b^2 - a^2$ must be even, but according to Proposition 2, the same is valid and for $c^2 + b^2 + a^2$, which implies that $c^2 + b^2$ must be even too.

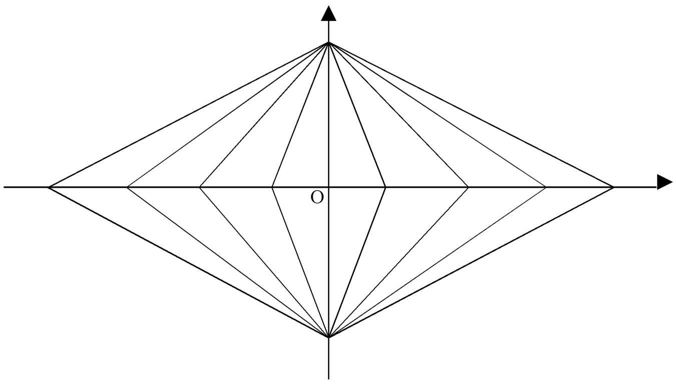
5. A kind of qualitative classification of Diophantian triangles. A classification of different types Diophantian, but non-Pythagorean, triangles can be given with the help of the notion of Pythagorean rectangle. This is a rectangle with an inscribed triangle for which the supplementary area is covered by Pythagorean triangles. The sketches below illustrate our idea.



We know different concrete examples of the above exposed sketches. Their uniqueness is conjectured in [2].

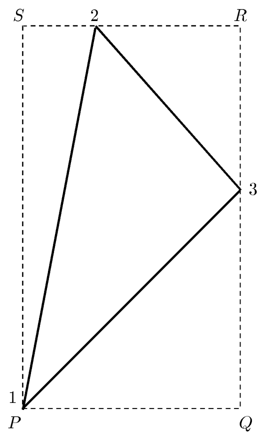
6. Symmetries, Diophantian carpets.

In general Diophantian figures don't admit symmetries. However some of them are symmetric with respect to axes. Example is given below.



In the next we collect different Diophantian figures and Diophantian carpets. By definition *Diophantian carpet* is a figure which is equipped with a triangulation by Diophantian triangles, but it is not itself a Diophantian figure.

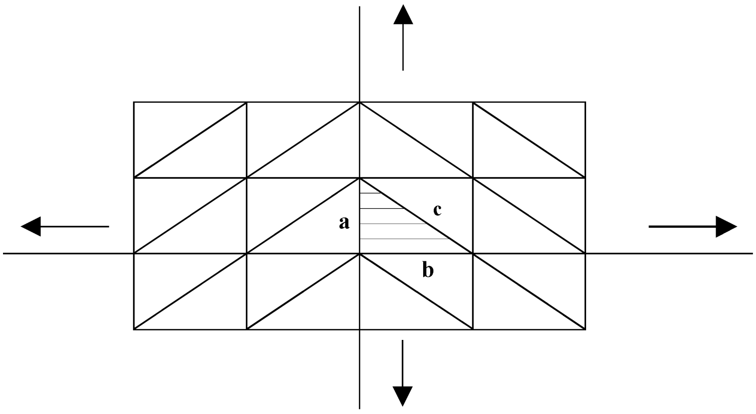
I. Examples of non-symmetric Diophantian figures.



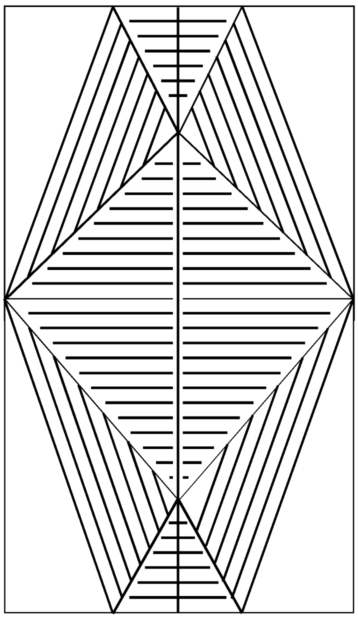
In the above sketch the triangles $PQ3$, $3R2$, $2S1$ are Pythagorean, and 123 is a Diophantian one. Clearly, the triangulation of the rectangle $PQRS$ is composed by four Diophantian triangles. So, $PQRS$ is a Diophatian carpet.

II. Infinite Diophantian carpets.

A simplest infinite Diophantian carpet is defined by a Pythagorean triple (a, b, c), where a and b are considered as legs with the condition $\pi(a) = \pi(b) = 1$.



III. Examples of a symmetric Diophantian carpet.



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ДИОФАНТОВИ ФИГУРИ И ДИОФАНТОВИ КИЛИМИ

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В тази статия излагаме конструкции на диофантови фигури получени с помощта на компютърни експерименти и геометрични разглеждания. Понятието Диофантова фигура бе въведено в [1] и развито в [2].