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GALERKIN SPECTRAL METHODS FOR HIGHER-ORDER BOUNDARY VALUE PROBLEMS ARISING IN FLUID MECHANICS*

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We develop Galerkin spectral technique for solving boundary value problems arising in natural convection. They consist of a fourth-order b.v.p. for the stream function coupled to a second-order b.v.p. for the temperature. As a basis are used the set of so-called Beam functions introduced by Lord Rayleigh and the set of Fourier functions. The formulas for the cross expansions between the two sets are derived and a Galerkin spectral algorithm is created. A featuring example is solved and the issues of the rate of convergence and truncation error are clarified.

1. Introduction. The treatment of boundary conditions is of crucial importance in numerical modelling of physical problems. Very often a difference or finite-element scheme can be rendered useless by a non-adequate approximation of the boundary conditions. This is especially true when the operators are of higher order and require more than one boundary condition. A typical example is furnished by the convective flows of viscous liquids, where a fourth order b.v.p. for the stream function is coupled to a second-order b.v.p. for the temperature. The application of standard elliptic solvers is a nontrivial matter for such flows.

There is a compelling need to develop fast spectral algorithms allowing a rapid interrogation of parameter space in order to discover and understand mechanisms of flow, and instability. The performance of a spectral method depends heavily on the type of the basis system. An elucidating discussion on the performance of different sets of functions can be found in the encyclopedic book of Boyd [2].

A way to overcome the difficulties connected with the boundary conditions is to use spectral methods with basis functions that satisfy the boundary conditions. A spectral expansion with a basis set which does not satisfy all of the boundary conditions would exhibit very poor convergence near the boundaries. For instance, the Fourier functions satisfy only one of the b.c. at each boundary. For the plane-parallel viscous flows (e.g., Poisuille flows) the so-called "beam" functions can be used to this end. They were proposed by Lord Rayleigh [9] to solve the vibration problems of elastic beams and since applied to different problems from fluid dynamics (see [3]).

The application of the *Beam*-Galerkin method to Poiseuille flow is already developed (see [8], [4]). We go a step further here and consider the generic boundary value problem of convective flows of viscous liquids.

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2. Posing the problem. Consider the 2D flow in a vertical slot with a linear vertical temperature gradient, differentially heated walls, and harmonic gravity modulations. The problem definition is well-described in the literature (refer to [1], [6]), and the notation we use is standard:

$$x = \frac{x^*}{L} - 1,\ y = \frac{y^*}{L},\ \omega = \omega^* \frac{L^2}{\kappa},\ t = t^* \omega^*,\ \psi = \frac{\psi^*}{\nu},\ \theta = \frac{T^*}{\delta T} + x - \tau_B y$$

where: ν is the kinematic viscosity, κ the thermal diffusivity, $2L$ the width of the slot, and δT the horizontal temperature difference. The asterisk denotes dimensional variables, while the same notation without an asterisk stands for the respective dimensionless quantity. The Rayleigh number Ra, the Prandtl number Pr, and stratifications parameter, γ, are defined as:

$$Ra = \frac{\beta g_0 \delta T L^3}{\nu \kappa},\ Pr = \frac{\nu}{\kappa},\ 4\gamma^4 = \tau_B Ra.$$

where β is the coefficient of thermal expansion of the liquid, g_0 the mean gravity, ε the dimensionless amplitude of gravity modulations, ω the dimensionless frequency, and τ_B is the dimensionless vertical temperature gradient.

For a comprehensive investigation of the 2D flow we refer the reader to [5]. The problem also admits a plane-parallel solution of the form $\Psi(x,t), \Theta(x,t)$ for which the governing system reduces to the following:

$$\frac{\omega}{Pr} \frac{\partial^3 \Psi}{\partial t \partial x^2} = -Ra \left[1 + \frac{\partial \Theta}{\partial x}\right] [1 + \varepsilon \cos(t)] + \frac{\partial^4 \Psi}{\partial x^4} \tag{1}$$

$$\omega \frac{\partial \Theta}{\partial t} = \tau_B \frac{\partial \Psi}{\partial x} + \frac{\partial^2 \Theta}{\partial x^2} \tag{2}$$

with the boundary conditions

$$\Psi = \frac{\partial \Psi}{\partial x} = \Theta = 0 \quad \text{for} \quad x = \pm 1\,, \tag{3}$$

The 1D flow was first treated in [6] where different regimés of flow were studied. The parametric bifurcation of the 1D solutions was studied in detail in [5] by means of a fully implicit difference scheme and a related 1D problem in [10]. As far as we are concerned here with developing of a new technique we should avoid the unnecessary complications connected with the oscillatory nature of the solutions in time. For this reason we focus our attention on a steady problem (ODE) with qualitatively similar structure of the spatial operators.

The higher-order coupled b.v.p. for an ODE system which retains all of the important terms in the full-fledged unsteady problem for the thermal convection in a vertical slot:

$$\frac{d^4 \Psi}{dx^4} = Ra \left[-1 + \frac{d\Theta}{dx}\right] + \frac{1}{Pr} \frac{\partial^2 \Psi}{\partial x^2}, \qquad \Theta - \tau_B \frac{d\Psi}{dx} = \frac{d^2 \Theta}{dx^2}, \tag{4}$$

$$\Psi = \Psi_x = \Theta = 0, \quad \text{for} \quad x = \pm 1.$$

We find the above system generically representative of the problem under consideration because it retains the second spatial derivatives. In a sense, it can be considered as a simplification of an Euler time-stepping scheme with time increment equal to one.

3. The spectral method. Consider the Sturm-Liouville problem

$$\frac{d^4u}{dy^4} = \lambda^4 u\,, \quad u = \frac{du}{dy} = 0\,, \quad \text{for} \quad x = \pm 1\,. \tag{5}$$

The nontrivial solutions (eigen-functions) of this problem are given by

$$s_m = \frac{1}{\sqrt{2}}\left[\frac{\sinh \lambda_m x}{\sinh \lambda_m} - \frac{\sin \lambda_m x}{\sin \lambda_m}\right], \qquad \text{cotanh}\ \lambda_m - \text{cotanh}\ \lambda_m = 0\,. \tag{6}$$

$$c_m = \frac{1}{\sqrt{2}}\left[\frac{\cosh \kappa_m x}{\cosh \kappa_m} - \frac{\cos \kappa_m x}{\cos \kappa_m}\right], \qquad \tanh\ \kappa_m + \tan\ \kappa_m = 0\,. \tag{7}$$

These functions were introduced by Lord Rayleigh to solve problems arising in beam theory and are sometimes called "beam" functions. A major step in the advancement of the application of the beam functions to fluid-dynamics problems was made by Poots [8]. The magnitudes of the different eigen values can be found in most of the above cited works from the literature. Chandrasekhar [3] derived their counterparts for problems with cylindrical symmetry.

The derivatives can be expressed in series with respect to the system as follows

$$c'_n = \sum_{m=1}^{\infty} a_{nm} s_m\,, \qquad a_{nm} = \frac{4\kappa_n^2 \lambda_m^2}{\kappa_n^4 - \lambda_m^4} \tag{8}$$

$$s'_n = \sum_{m=1}^{\infty} \bar{a}_{nm} c_m\,, \qquad \bar{a}_{nm} = \frac{4\kappa_m^2 \lambda_n^2}{-\kappa_m^4 + \lambda_n^4} \tag{9}$$

$$c''_n = \sum_{m=1}^{\infty} \beta_{nm} c_m, \quad b_{nm} = \begin{cases} \dfrac{4\kappa_n^2\kappa_m^2}{\kappa_m^4 - \kappa_n^4}\left(\kappa_m \tanh\ \kappa_m - \kappa_n \tanh\ \kappa_n\right) & m \neq n \\ \kappa_n \tanh\ \kappa_n - (\kappa_n \tanh\ \kappa_n)^2 & m = n \end{cases} \tag{10}$$

$$s''_n = \sum_{m=1}^{\infty} \bar{\beta}_{nm} s_m, \quad \bar{b}_{nm} = \begin{cases} \dfrac{4\lambda_n^2\lambda_m^2}{\lambda_n^4 - \lambda_m^4}\left(\lambda_n \text{cotanh}\lambda_n - \lambda_m \text{cotanh}\lambda_m\right) & m \neq n \\ \lambda_n \text{cotanh}\lambda_n - (\lambda_n \text{cotanh}\lambda_n)^2 & m = n \end{cases} \tag{11}$$

Formulas expressing the third derivatives and the products of two *beam* functions into series with respect to the system wcan be found in [4], [7].

For the convective problem under consideration the difficulties arise from the fact that the boundary value problem for temperature function is of second order which means that the system of beam functions is not suitable for expanding the temperature field. It is clear that the best suited to the task system are the trigonometric *sines* and *cosines*. Hence we need to develop expressions for expanding the beam functions into trigonometric functions and vice versa:

$$\sin l\pi x = \sum_{k=1}^{\infty} \sigma_{lk} s_k(x), \qquad \sigma_{lk} = \frac{2\sqrt{2} l\pi (\lambda_k)^2 (-1)^l}{l^4\pi^4 - \lambda_k^4} \tag{12}$$

$$\cos l\pi x = \sum_{k=1}^{\infty} \chi_{lk} c_k(x), \qquad \chi_{lk} = \frac{2\sqrt{2}\kappa_k^3 (-1)^{l+1} \tanh\ \kappa_k}{l^4\pi^4 - \kappa_k^4} \tag{13}$$

$$c_n(x) = \sum_{l=1}^{\infty} \hat{\chi}_{nl} \cos l\pi x, \qquad \hat{\chi}_{nl} = \frac{2\sqrt{2}\kappa_n^3(-1)^{l+1} \tanh\ \kappa_n}{l^4\pi^4 - \kappa_n^4} \tag{14}$$

$$s_n(x) = \sum_{l=1}^{\infty} \hat{\sigma}_{nl} \sin l\pi x, \qquad \hat{\sigma}_{nl} = \frac{2\sqrt{2}l\pi(\lambda_n)^2(-1)^l}{l^4\pi^4 - \lambda_n^4} \tag{15}$$

to which has to be added also the expansion of unity in series of c_n functions.

$$1 = \sum_{k=1}^{\infty} h_k c_k(x), \quad h_k = \int_{-1}^{1} c_k(x)\mathrm{d}x = \frac{2\sqrt{2}\tanh\ \kappa_k}{\kappa_k} \tag{16}$$

We point out that the convergence when expanding *unity* and $cos(l\pi x)$ into c_k series is first order k^{-1} (see (13)) due to the fact that it does not satisfy both b.c. for the beam functions. It satisfies the condition on the derivatives but fail to satisfy the conditions on the function itself. Clearly, the situation with the $sin(l\pi x)$ is better and the rate of convergence is of second order k^{-2} (see (12)) because the *sine* functions satisfy the boundary conditions on the functions and the disagreement is more subtle since the conditions on the first derivative are not satisfied. The situation with the expansions of s_x and c_k in Fourier series is reversed. The order of convergence for c_k is l^{-4} (see (14)), and for s_x is l^{-3} (see (15)). As it will be shown in what follows, this property is of crucial importance for the overall rate of convergence.

4. Results and discussion. Because of the obvious symmetry of the boundary value problem under consideration we can seek a solution in which the stream function is even and the temperature is odd function. Then we can develop the sought functions into the following series

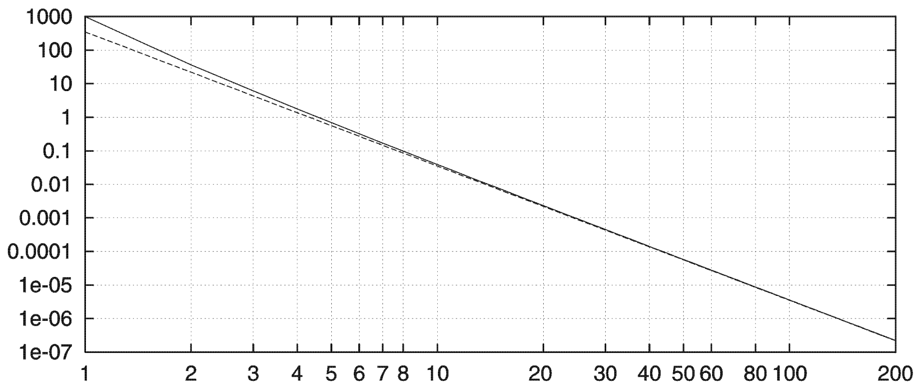
$$\Psi(x,t) = \sum_{k=1}^{K} p_k c_k(x), \qquad \Theta(x,t) = \sum_{k=1}^{K} d_k \sin(k\pi x), \tag{17}$$

Upon introducing these expansions into (1), (2) and making use of the above compiled formulas, an algebraic system for the coefficients d_k and p_k is derived

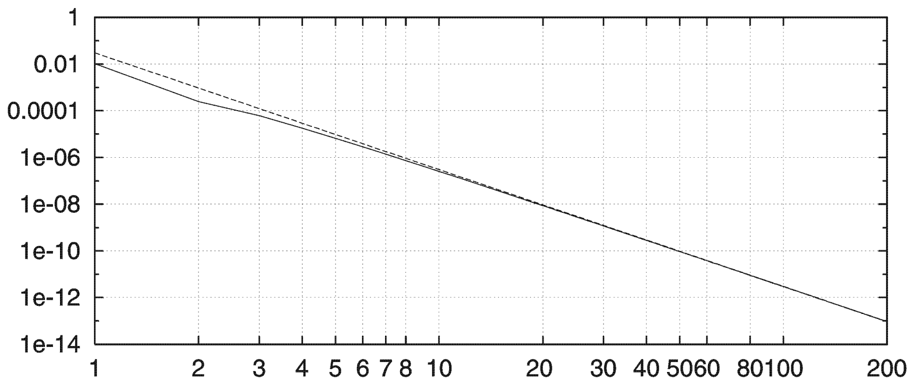
$$-\kappa_i^4 p_i + \frac{1}{Pr}\sum_{j=1}^{N} p_j\beta_{ij} = -Ra\left[\sum_{m=1}^{N} d_m \frac{m\pi 2\sqrt{2}(-1)^{m+1}\kappa_i^3 \tanh\ \kappa_i}{m^4\pi^4 - \kappa_i^4} - \frac{2\sqrt{2}\tanh\ \kappa_i}{\kappa_i}\right] \quad \text{for } i = 1, \ldots, N. \tag{18}$$

$$(1 + l^2\pi^2)d_l = \tau_B \sum_{n=1}^{N}\sum_{m=1}^{N} p_n \frac{8\sqrt{2}\kappa_n^2\kappa_m^2 l\pi(-1)^l}{(\kappa_n^4 - \kappa_m^4)(l^4\pi^4 - \kappa_m^4)} \quad \text{for } l = 1, \ldots, N. \tag{19}$$

The results for the coefficients p_i and d_l are presented in Fig. 1. The peculiar finding is that the rate of convergence for Θ is algebraic of fifth order while the rate for Ψ is one order lower (fourth order). The analytical explanation of this phenomena will be the object of a separate study. Here it suffice to mention that the off-diagonal elements in



Solid line: p_i; Dashed line: $p_i = 350i^{-4}$



Solid line: d_i; Dashed line: $d_i = 0.03i^{-5}$

Fig. 1. The rate of convergence for the coupled system for $Ra = 6000$, $Pr = 1$ and $\tau_B = 0.001$. The upper panel shows the spectral coefficient for function Ψ; the lower panel shows Θ.

(18) can degrade the rate of convergence, while in the equation (19) for Θ no off-diagonal elements are present and the convergence is of fifth order.

The fourth order for the rate of convergence means that a number of terms $N = 100$ is fully adequate to obtain results with very high precision 10^{-8}.

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СПЕКТРАЛНИ МЕТОДИ ОТ ТИП ГАЛЕРКИН ЗА ГРАНИЧНИ ЗАДАЧИ ОТ ВИСОК РЕД, ВЪЗНИКВАЩИ В МЕХАНИКАТА НА ФЛУИДИТЕ

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Предложен е нов спектрален метод от тип Галеркин за решаване на гранични задачи, възникващи при естествена конвекция. Тези задачи се свеждат до решаване на гранична задача от четвърти ред за потока и на гранична задача от втори ред за температурата. В основата на предложения подход са така наречените Beam functions, въведени от лорд Rayleigh, както и функциите на Fourier. Решен е един типичен пример, като са изследвани скоростта на сходимост на предложения алгоритъм и грешките от закръгляване.