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ON SOME IMPRECISE GEOMETRICAL COMPUTATIONS BASED ON MORPHOLOGICAL TECHNIQUES

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This work studies some problems arising in image processing, computational geometry and robot guidance due to uncertainty and imprecision in the envionmental model and the input data. More precisely, based on so-called Epsilon-geometry approach combined with morphological techniques we find ε-connected components of a given object and analyse the opportunities for approximations of the skeleton of the object.

1. Imprecise computations In many practical tasks we operate with imprecise or uncertain data, especially when this data comes as an output from some measuring instrument. This problem appears in image processing because of the distortion effect, and often because of nonprecise calibration of the camera. Therefore lots of work have been done for the development of algorithms which give reliable results operating with imprecise data.

The basic idea of interval mathematics is that the set-theoretic intervals are the consistent context for numerical computing. However, for example, in solving equations with interval coefficients algebraic completeness of the interval operations is required which is not guaranteed by the ordinary set-theoretic interval operations. One possible way to correct the situation is to use modal interval analysis [3]. Modal interval analysis define intervals starting from the identification of real numbers with the families of predicates they satisfy or not. Let us consider the family of set-theoretic intervals $I(\mathbf{R})$ and the set of predicates on $\mathbf{R}$ denoted by $Pred(\mathbf{R}) = \{\mathtt{P}|\mathtt{P} : \mathbf{R} \mapsto \{\mathtt{0},\mathtt{1}\}\}$. A modal interval is an element of the cartesian product $(I(\mathbf{R}), \{\forall, \exists\})$. Then a modal interval with first element $[a, b]$ $(a \leq b)$ is reffered to as *proper* if it is characterized by those predicates true for all points (the second element of the couple is the quantifier $\forall$), while it is called *improper* if it is characterized by those predicates true for at least one point in $[a, b]$ (the second element of the couple is the quantifier $\exists$). Then for $a \leq b$ the improper interval with first part $[a, b]$ is denoted simply by $[b, a]$ and is said to be of negative direction. Detailed description of the generalized interval arithmetic operations can be found in [3] and [11].

The Epsilon-Geometry framework follows the idea of modal interval analysis for solving geometric problems. It is based on a general model of imprecise computations, which includes rounded-integer and floating-point arithmetic as special cases [5]. The Epsilon-Geometry framework defines the notion of an *epsilon predicate* as a means for creating approximate tests. Let $\mathcal{O}$ be a set of objects in a space supplied with a metric, or pseudometric d. Let $\mathtt{P}$ be a predicate defined on $\mathcal{O}$. Then for any $X \in \mathcal{O}$ and any $\varepsilon > 0$, let us define an epsilon version of $\mathtt{P}$ [4], [5].Say that $\varepsilon-\mathtt{P}(X) = \mathtt{true}$ if and only if $\mathtt{P}(X') = \mathtt{true}$ for **some** $X',\ d(X,X') \le \varepsilon$. $(-\varepsilon) - \mathtt{P}(X) = \mathtt{true}$ if and only if $\mathtt{P}(X') = \mathtt{true}$ for **all** $X',\ d(X,X') \le \varepsilon$. For instance a polygon is said to be $(-\varepsilon)$-convex if it remains convex under any perturbation to its vertices in disks with radius $\varepsilon > 0$.

2. Basic morphological operations. Here and henceforth, by $\mathcal{P}(X)$ we denote the power set of X, i.e. the family of all subsets of the set X. Then every translation-invariant dilation is represented by the standard Minkowski addition: $\delta_A(X) = A \oplus X = X \oplus A$, and its adjoint erosion is given by Minkowski subtraction: $\varepsilon_A(X) = X \ominus A$ [6]. Then closing and opening of A by B are defined as $A \bullet B = (A \oplus B)\ominus,\ A \circ B = (A \ominus B) \oplus B$, These operations are referred to as *classical* or *binary morphological operations.*

3. Connectivity and epsilon-geometry.

3.1. Connectivity on complete lattices. In mathematics, the notion of connectivity is formalized in the topological framework in two different ways. First, a set is called to be connected when it cannot be partitioned as a union of two open, or two closed sets. In practice, it is more suitable to work with the so-called *arcwise connectivity.* A set X is said to be arcwise connected when for every two distinct points a and b from X there exists a continuous curve joining a and b and lying entirely in X. Arcwise connectivity is more restrictive then the general one. It is not difficult to show that any arcwise connected set in $\mathbf{R}^n$ is connected. The opposite is not true, elsewhere we consider only open sets in $\mathbf{R}^n$.

Arcwise connectivity is widely used in robot motion planning. The motion planning task is to find a path, i.e. a continuous sequence of collision-free configurations of the robot (or any moving agent referred to as a robot), connecting two arbitrary input configurations (the start configuration q_b and the final configuration q_e) whenever such a path exists, or indicate that no such path exists. The negative result means that the query points q_b and q_e lie in different connected components of the free configuration space. In this case it is evident the profit of studying approximate connectivity – if there are uncertainties in the robot metrics and control parameters, the robot may collide with the obstacles when moving through narrow passages in the workspace [7], [1]. If the geometric models of the robot and the workspace are imprecise, the approximate connectivity approach could be useful in practice, especially when the path planner captures the connectivity of the robots configuration space by building a probabilistic roadmap [7], a network of simple paths connecting points picked in random in this space, or when using wave wave-propagation algorithm [1].

In image analysis, several notions of digital connectivity has been introduced. Usually, they exploit the definition of arcwise connectivity in a discrete way – depending on the regarded neighbourhood relation (4-square, 8-square, hexagonal, etc.) [13]. Following the the works of Serra [13] and Heijmans [6], an abstract connectivity framework, suited mainly for analysis and processing of binary images has been developed. It is strongly related with the mathematical morphology concepts. The base concept is the *connectivity class*:

Definition 3.1. *Let E be an arbitrary set. A family $\mathcal{C}$ of the subsets of E is called a connectivity class if the following properties hold:*

1. $\emptyset \in \mathcal{C}$ *and* $\{x\} \in \mathcal{C}$ *for every* $x \in E$ *;*
2. *If* $E_i \in \mathcal{C}$ *for* $i \in I$ *and* $\bigcap_{i\in I} E_i \neq \emptyset$ *, then* $\bigcup_{i\in I} E_i \in \mathcal{C}$.

Also it is sometimes useful to impose additional conditions, such as *translation invariance* of the connectivity. It hasn't been imposed in the previous works [13, 12]. This condition can be replaced by more general one, namely *affine invariance of connectivity.* However, in our work it is sufficient to work only with translation-invariant operators.

Given a connectivity class in a universal set E we can define the maximal connected component of a set $A \subseteq E$ containing a particular point x:

$$\gamma_x(A) \;=\; \bigcup\{C \in \mathcal{C}\,|x \in C \text{ and } C \subseteq A\}$$

Then it can be proved easily that [6]:

— For every $x \in E \quad \gamma_x$ is a morphological opening in E;

— $\gamma_x(\{x\}) = \{x\}$.

— either $\gamma_x(A) = \gamma_y(A)$ or $\gamma_x(A) \cap \gamma_y(A) = \emptyset$.

— $\bigcup_{x \in E} \gamma_x(A) = A$.

It is easy to demonstrate that $X \in \mathcal{C}$ if and only if for every two points $x, y \in X$ it follows that $\gamma_x(X) = \gamma_y(X)$.

It is easy to demonstrate the following result:

Theorem 3.2. *If X and A are connected with respect to the connectivity class $\mathcal{C}$, then $X \oplus A$ is connected with respect to $\mathcal{C}$ as well.*

This theorem is a straightforward generalization of Theorem 9.59 from [6], where only arcwise connectivity is considered.

Finding maximal connected components is found to be useful in computer-aided tomography for separating the different tissues on the image.

Let $\mathcal{S}$ be a binary relation between the subsets of a universal set E, i.e.

3.2. Epsilon-geometry approach. Let γ_x be the connectivity openings in E associated with the path – connectivity (the usual arcwise connectivity in the continuous case $E = \mathbf{R}^n$, or its discrete analogs as mentioned above). Let $K \subseteq E$ be a connected structuring element which contains the origin. Then we can define another family of openings: $\gamma'_x(A) = A \cap \gamma_x(A \oplus K)$ for every point $x \in A$. These openings are connectivity openings with respect to the connectivity class $\mathcal{C}' = \{D \subseteq \mathbf{R}^n \,|\, D \subseteq C \subseteq D \oplus K$ for some connected set $C \in \mathcal{C}\}$, (see Example 9.62 from [6]). Then having as a base the usual path connectivity in its continuous or discrete versions, we can introduce the notion of *epsilon connectivity.*

Definition 3.3. *A set $X \subseteq E$ is called ε-connected if it is connected with respect the connectivity class $\mathcal{C}'$ when $K = B_\varepsilon(0)$ is the disk with radius ε centered at the origin.*

An example of this notion is given on figure 1.

If P and Q are non-empty compacts in $\mathbf{R}^n$, then

$$\text{(1)} \qquad \operatorname{dist}(P, Q) \;=\; \inf\{\varepsilon \,|\, Q \subseteq P \oplus B_\varepsilon(0)\,,\, P \subseteq Q \oplus B_\varepsilon(0)\}.$$

is known as *Hausdorff distance* between P and Q. In practice, in all image processing and robot control tasks we work with compact sets. Therefore our definition of epsilon-connectivity is correct, since for every set X' from $\mathcal{C}'$ there exists a set X from $\mathcal{C}$ such that $\operatorname{dist}(X, X') \leq \varepsilon$. Here, the notion of $(-\varepsilon)$-connectivity is not useful, since we should have a connectivity class, which elements must have the property, that all sets at a distance not greater than ε should be path–connected for $\varepsilon > 0$.

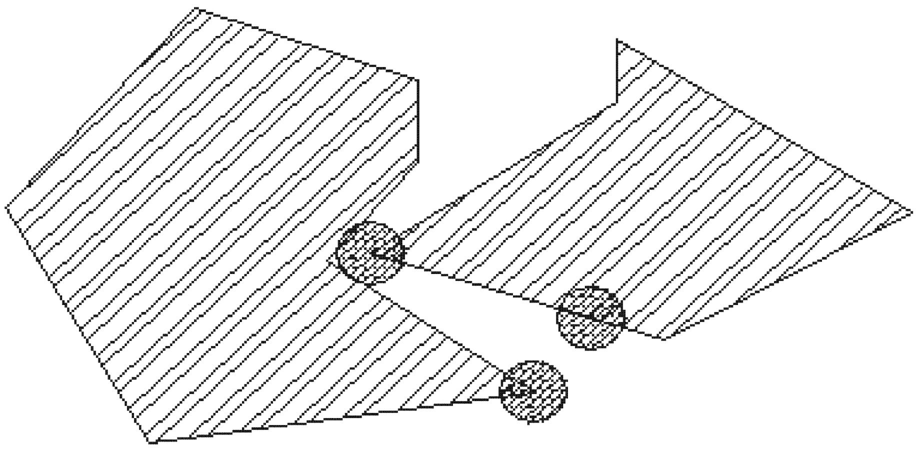

Fig. 1. The hatched region becomes ε-path-connected addding the disk, with ε equal to the radius of the disk

4. Set skeletons and its epsilon–construction.

4.1. Set skeleton–definition. Let A be a compact (closed and bounded) set in $\mathbf{R}^n$. *The skeleton* of A is defined as the set of the centres of maximal balls inscribed in A, i.e.

$$SK(A) = \{x \in \text{int}\,(A) \mid (\exists r > 0)$$
$$(\forall r' > 0)(\forall x' \in \mathbf{R}^n)[(B_r(x) \subseteq B_{r'}(x') \subseteq A) \Rightarrow \ (x = x' \,\&\, r = r')]\}.$$

Here and henceforth int (A) denotes the interior of the set A, while cl (A) and ∂A denote the topological closure and the border of A, respectively. For the main topological properties of the skeleton see [9].

4.2. Medial axis and its relations with the skeleton. Let us consider an n-dimensional linear space M over the field of natural numbers $\mathbf{R}$ or over the ring of integers $\mathbb{Z}$, called domain. Let $K \subseteq M$ be bounded centrally symmetric structuring element.

Let A be an arbitrary subset of M. Consider the function $d[A,K] : M \to \mathbb{N}$ defined as follows:

$$d[A,K](x) = \begin{cases} \max\{m \in \mathbb{N} \,|x \in A \setminus_{m-1} K\} & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Denote $A \setminus (K^{\oplus n}) = A \setminus (\underbrace{K \oplus \ldots \oplus K}_{n-times})$, for $n \geq 1$ and $A \ominus_0 K = A$. This function is called *distance transform of A by the structuring element K* [6].

Let us define *medial axis of the set A by the structuring element K* in the following way: $MA(A,K) = \{x \in A \,|\, d[A,K](x) \geq d[A,K](y)$ for every $y \in K_x\}$, i.e. the medial axis $MA(A,K)$ is the collection of extremal points for the distance transform.

Theorem 4.1 *Let A be a compact set from $\mathbf{R}^n$ with nonempty interior. Then $SK(A) \subseteq MA(A, B_\varepsilon(0))$ for any positive ε.*

Theorem 4.2. *Let A be a compact set from $\mathbf{R}^n$ with nonempty interior. Then $MA(A, B_\varepsilon(0)) \cap \mathrm{int}\,(A) \subseteq SK(A) \oplus B_\varepsilon(0)$*

The upper lemmas and theorems are proved in [10].

Consider an example of an object – two big enough disks connected with a line segment. Its skeleton consists only of the centres of the two disks, while the medial axis by $B_r(0)$ is made of two disks with radius r centered at the centres of the big disks together with the connecting segment. Therefore the stronger then the statement of the last theorem inclusion $MA(A, B_r(0)) \subseteq SK(A) \oplus B_r(0)$ is not true.

Having in mind the definition of Hausdorff distance, our result shows that the approximation is one-side in general. Therefore, the medial axis is not an approximation of the skeleton in the terms of ε-geometry. Therefore we found the reason why in some practical applications, the approach using the medial axis construction leads ocasionally to non-connected skeleton for a connected set. Alternatively, for a boundary represented object, one can use an approximate construction of its skeleton by using the Voronoi diagram of a discrete sample set over its boundary [2] to obtain a thin set which do not break. In the case we know all the pixels of the region we want to skeletize, we may use a heuristic aprroach based on consecutive morphological thinnings by suitably chosen structuring elements, which often leads to satisfactory results [6].

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ВЪРХУ НЯКОИ НЕТОЧНИ ГЕОМЕТРИЧНИ ПРЕСМЯТАНИЯ ОСНОВАНИ НА МОРФОЛОГИЧНИ ТЕХНИКИ

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В настоящата работа се разглеждат задачи от обработката на изображения, планирането на движения на роботи и изчислителната геометрия, и по-точно проблемите в тях, възникващи от неточности в моделите и входните данни. Използвайки методи от математическата морфология в комбинация с т. нар. Епсилон геометрия, намираме ε–свързаните компоненти на даден обект, а също така и възможностите за приближена скелетизация.