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MODELLING CHAOS-HYPERCHAOS TRANSITION IN COUPLED RÖSSLER SYSTEMS

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Many of the nonlinear high-dimensional systems have hyperchaotic attractors. Typical trajectory on such attractors is characterized by at least two positive Lyapunov exponents. We provide numerical evidence that chaos-hyperchaos transition in continuous systems can be characterized by the set of infinite number of unstable periodic orbits embedded in the attractor as it was previously shown for the case of coupled discrete maps.

Unstable periodic orbits (UPO's) constitute the most fundamental blocks of a chaotic system [1]. Theoretically, the infinite number of UPO's emmbedded in a chaotic set provides the skeleton of the attractor and allows the estimation of many dynamical invariants such as the natural measure, the spectra of Lyapunov exponents, the fractal dimension in the fundamental way [2]. Recently UPO's have been used in the description of higher-dimensional dynamical phenomena such as blowout bifurcation [3] – [6] and chaos-hyperchaos transition (i.e. transition from the attractor characterized by one positive Lyapunov exponent to the attractor characterized by at least two positive exponents) [7]. It has been shown that chaos-hyperchaos transition as well as blowout bifurcation is mediated by an infinite number of UPO's which become repellers in the neighborhood of the transition point. The simultaneous existence of UPO's with different number of unstable direction gives rise to the nonhyperbolicity known as unstable dimension variability and provides a possible dynamic mechanism for the smooth transition through zero of second Lyapunov exponent.

Up to now the description chaos-hyperchaos transition using UPO's has been performed only for the case of coupled discrete maps. In this paper we argue and provide numerical evidence that this description can be applied to the continuous dynamical systems (flows). We show that the balance of the appropriate weights of UPO's orbits with one unstable dimension and UPO's with at least two unstable dimension gives the approximation of chaos-hyperchaos transition point.

As an example consider the two identical symmetrically coupled Rössler systems

$$\dot{x}_1 = -x_2 - x_3,$$

$$\dot{x}_2 = x_1 + ax_2,$$

$$\dot{x}_3 = b + x_3(x_1 - c) + d(y_3 - x_3), \tag{1}$$

$$\dot{y}_1 = -y_2 - y_3,$$
$$\dot{y}_2 = y_1 + ay_2,$$
$$\dot{y}_3 = b + y_3(y_1 - c) + d(y_3 - x_3),$$

where $(x_1, x_2, x_3, y_1, y_2, y_3) \in \mathcal{R}^6$ are dynamical variables, a, b, c are constant system parameters and d is the coefficient of coupling. It is well-known that the Rössler system develops continuous chaos through period-doubling bifurcation cascade [8]. Since the Rössler system has a foundation in the kinematics of chemical reaction [9], it is natural to study the diffusive coupling of two such systems [10].

In our numerical studies we took the following parameter values $b = 2.0$, $c = 4.0$, $d = 0.25$ and consider a as a control parameter. With the increase of the control parameter a the system (1) reveals the transition to hyperchaos [10] with a smooth passing of the second Lyapunov exponent through zero. The variation of four Lyapunov exponents versus a (two other exponents are equal to zero) is shown in Fig. 1.

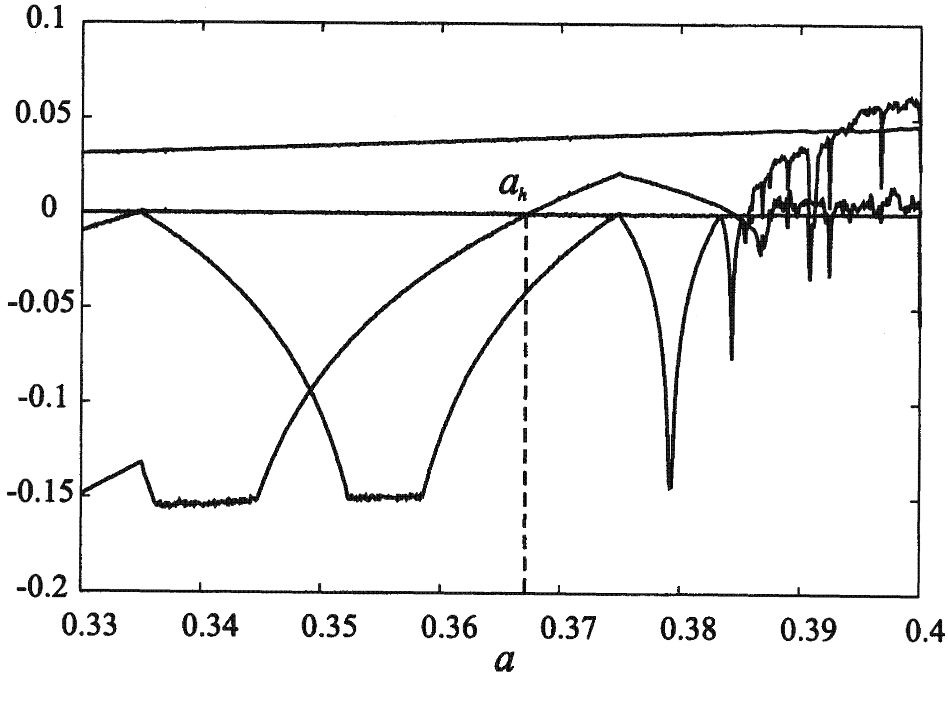

Fig. 1. The variation of four Largest Lyapunov exponents for the coupled Rössler system (1) for $d = 0.25$. The smooth transition to hyperchaos occurs at $a_h \approx 0.3673$

One can observe a typical smooth transition to hyperchaos (similar to this observed in [11], [12]) at $a \approx 0.3673$.

In the following, we try to investigate stability of low-periodic orbits embedded into the attractor of system (1) when it undergoes chaos-hyperchaos transition. In order to find and classify these orbits we use the Poincaré cross-section that is determined by the following normal vector $\mathbf{n}$=$(-3.75, 1.84, -6.48, 1.75, -2.09, -0.10)$ and a base point with coordinates $P = (1.72, 0.35, 3.40, -1.27, -2.29, 0.53)$, which was found to be more suitable for our system if we intend to consider only one of the symmetric attractors. Vector $\mathbf{n}$ was chosen along the flow (1) at point P.

Having found the admissible periods p, we applied the shooting method [14], [15] for locating periodic orbits of system (1) first getting the initial guess by iterating p-th power of the Poincaré map. The periods $p \leq 24$ were investigated. Let γ_i^p be a period-p orbit and ν_1, ν_2 be the largest multipliers of this orbit. Then $\lambda_1 = \ln(\nu_1)/T$, and $\lambda_2 = ln(\nu_2)/T$ be the first two largest Lyapunov exponents of the corresponding orbit of system (1) of a period T. It can be seen that the number of doubly unstable orbits increases and the number of cycles with one unstable multiplier decreases when one approaches chaos-hyperchaos transition point a_h.

More precise characteristic may be obtained by calculating weights $\Lambda_p^2(a)$ and $\Lambda_p^1(a)$ that correspond to doubly unstable cycles and cycles with one unstable multiplier, respectively, cf. [13], [7], [3]. They are defined as follows:

$$\Lambda_p^1 = \sum_{i=1}^{N_p^1} \rho(\gamma_i^p, p)\lambda_2'(\gamma_i^p), p)$$

$$\Lambda_p^2 = \sum_{i=1}^{N_p^2} \rho(\gamma_i^p, p)\lambda_2'(\gamma_i^p), p),$$

where N_p^1 and N_p^2 are the numbers of cycles that have one and two unstable directions, respectively. The cycle γ_i^p weight is [2]

$$\rho(\gamma_i^p, p) = p\frac{L_i^{-1}}{\sum\limits_{i=1}^{N_p^1+N_p^2} L_i^{-1}}, \tag{2}$$

where L_i is the product of the unstable multipliers of ith cycle, i.e. $L_i = \nu_1$ or $L_i = \nu_1\nu_2$ in the case of doubly unstable cycle. $\lambda_2'(\gamma_i^p), p) = \frac{1}{p}\ln\nu_2(\gamma_i^p, p)$ is the second Lyapunov exponent of a periodic orbit γ_i^p for the Poincaré map.

Following to [3], the quantity $\Delta\Lambda_p(a) = \Lambda_p^1 + \Lambda_p^2$ may serve as an approximation of a blowout bifurcation point for attractors in an invariant subspace. By the analogy to that approach and [7] we may approximate the chaos-hyperchaos transition point. Figure 2 shows variation of $\Delta\Lambda_p(a)$ with a for $p = 24$. For all parameter values from 1000 to 2000 fixed points of pth power of the Poincaré map were accumulated. Although we can not affirm that all periodic orbits up to period 24 are calculated, Fig. 2 shows clearly that approximately the same value of the weights Λ_p^1 and Λ_p^2 occurs near the chaos-hyperchaos transition point. This is specially visible as one considers the mean square approximation of $\Delta\Lambda_p(a)$ shown in broken line. Note that the fluctuations of $\Delta\Lambda_p(a)$ around zero before chaos-hyperchaos transition are due to the finite number of considered UPO's.

We have shown here that the transition from chaos to hyperchaos in higher-dimensional continuous dynamical system is a bifurcation that like in the case of coupled maps, is mediated by an infinite number of unstable periodic orbits. In the neighborhood of

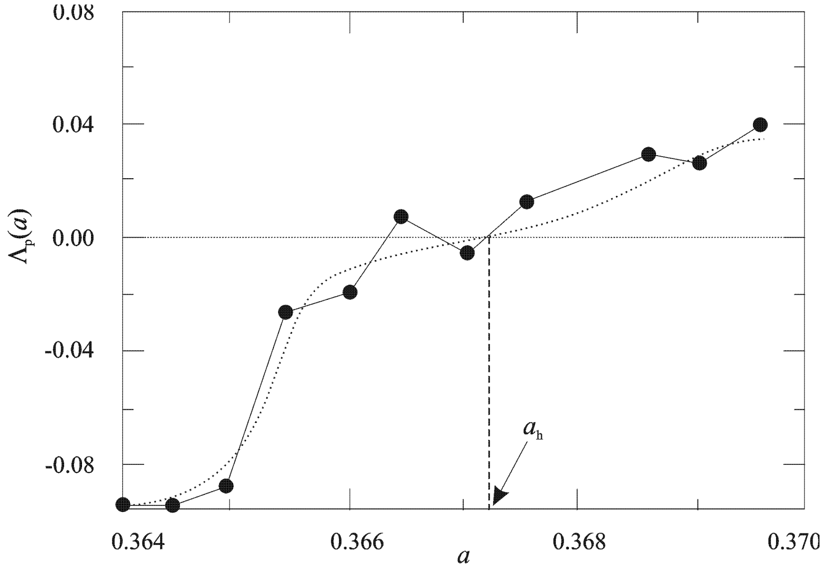

Fig. 2. Variation of $\Lambda_p(a)$ for $p = 24$. The approximation of the chaos-hyperchaos transition point by low-periodic orbits up to period $p = 24$, the broken line shows the mean-square approximation

the transition point one observes the co-existence of UPO's with one (saddles which are typical for 3-dimensional chaotic systems) and at least two unstable eigenvalues. This co-existence is responsible for the occurrence of nonhyperbolic behaviour known as unstable dimension variability and can explain the smooth passage through zero of the second Lyapunov exponent at chaos–hyperchaos transition point. In a continuous system despite the fact that it is impossible to determine all UPO's of the given period the balance of the appropriate weights of UPO's of different types can approximate the transition point in the control parameter space.

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МОДЕЛИРАНЕ НА ХАОС-ХИПЕРХАОС ПРЕХОД В ДВЕ СВЪРЗАНИ RÖSSLER-ОВИ СИСТЕМИ

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Много от многомерните системи имат хипер-хаотични атрактори. Типичните траектории за такива атрактори се характеризират чрез най–малко две положителни Ляпунови експоненти. Числено е показано, че хаос – хиперхаос преходът в непрекъснати системи може да се характеризира чрез едно безкрайно множество от неустойчиви периодични орбити, вложени в атрактора, аналогично на случая на двойка дискретни изображения.