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MEIJER'S G-FUNCTION: BULGARIAN TRACES FOR ITS USE IN SPECIAL FUNCTIONS, INTEGRAL TRANSFORMS AND FRACTIONAL CALCULUS *

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This is a short survey of the role of Meijer's G-function in the developments of the topics from the title, related to works of some Bulgarian mathematicians, and especially in author's studies since 1974. It is inspired by her recent visit to the Groningen University in Holland, the place where *Prof. Cornelis Simon Meijer* has studied and worked, and where now, the Bulgarian traces in this topic of Dutch origin have been emphasized. Another good provocation is the increasing attention that some foreign authors have recently paid to the history, development and actual applications to practical problems, of the so-called Mellin-Barnes type integrals giving rise to the most useful definition of the G-function. This generalized hypergeometric function happens to provide: solutions to wide classes of singular differential equations of arbitrary order with variable coefficients and to Volterra type integral equations; kernel-functions for a variety of linear integral and integro-differential operators of applied analysis; as well as a unification scheme for the other special functions of mathematical physics.
Key Words: Meijer's G-function, generalized hypergeometric functions, integral transforms of Laplace and Meijer, hyper-Bessel operators, fractional calculus.

1. Definition, basic properties and examples. *Meijer's G-function* has arisen in mathematical analysis in the attempts to give meaning to the symbols ${}_pF_q$ in the case $p > q+1$. The first Meijer's definition (1936, [19]) consists in considering, instead of a senseless ${}_pF_q$, a finite series of well defined generalized hypergeometric functions ${}_{q+1}F_{p-1}$ with $q+1 \le (p-1)+1$. This coincides, in essence, with the definition of the so-called E-function, introduced independently by McRobert (1938, [18]), in the form

$$E(p;\alpha_r:q;\beta_s:z) = \sum_{r=1}^{p} \frac{\prod_{s=1}^{p}{}'\,\Gamma(\alpha_s-\alpha_r)}{\prod_{t=1}^{q}\Gamma(\beta_t-\alpha_r)}\,\Gamma(\alpha_r)\,z^{\alpha_r}$$

$$\times\ {}_{q+1}F_{p-1}\left[\begin{matrix}\alpha_r,\alpha_r-\beta_1+1,\dots,\alpha_r-\beta_q+1;\\ \alpha_r-\alpha_1,\dots,*,\dots,\alpha_r-\alpha_p+1;\end{matrix}\ (-1)^{p+q}z\right], \tag{1}$$

where $|z|<1$, if $p=q+1$, $'$ in the product means that the multiplier $\Gamma(\alpha_r-\alpha_r)$ is omitted, and $*$ in F denotes that the parameter $\alpha_r-\alpha_r+1$ is omitted.

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Later on, Meijer (1941-1946, [21], [22]) has replaced this definition by more suitable one, in terms of Mellin-Barnes type integrals:

$$G^{m,n}_{p,q}\left[z\left|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right.\right]:=G^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_j)_1^p\\ (b_k)_1^q\end{matrix}\right.\right]$$

$$(2)\qquad =\frac{1}{2\pi i}\int_{\mathcal{L}}\frac{\prod_{k=1}^{m}\Gamma(b_k-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{k=m+1}^{q}\Gamma(1-b_k+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}z^s ds,\quad z\neq 0.$$

The integers $0\leq m\leq q, 0\leq n\leq p$ are known as orders of the G-function, and for the conditions on the parameters $a_j, j=1,\dots,p$; $b_k, k=1,\dots,q$ and the three possible types of contours $\mathcal{L}$ in $\mathbf{C}$, the details can be seen in any of the books [7], [14], [17], [26], [10]. Definition (2) gives greater freedom of choice for the values of p and q and allows the use of the powerful tools of complex analysis, especially the theory of contour integration and the residue theorem (that leads again to expressions like (1)).

Integrals (2), defining a wide class of *generalized hypergeometric functions*, incorporate as special cases almost *all the known special functions of mathematical physics*, including the ${}_pF_q$-functions

$${}_pF_q\left[\begin{matrix}\alpha_1,\dots,\alpha_p\\ \beta_q,\dots,\beta_q\end{matrix}\ |z\right]=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\dots(\alpha_p)_k}{(\beta_1)_k\dots(\beta_q)_k}\frac{z^k}{k!}$$

$$(3)\qquad =\frac{\prod_{k=1}^{q}\Gamma(b_k)}{\prod_{j=1}^{p}\Gamma(a_j)}G^{1,p}_{p,q+1}\left[-z\left|\begin{matrix}1-\alpha_1,\dots,1-\alpha_p\\ 0,1-\beta_1,\dots,1-\beta_q\end{matrix}\right.\right],$$

and all their particular cases, as well as the *basic elementary functions.* Thus, we can recognize as G-functions, the *following examples*: the Gauss function ${}_2F_1(a,b;c;z)$ as a $G^{1,2}_{2,2}(-z)$; the confluent hypergeometric functions of Kummer and Tricomi as $\Phi(a;c;z)\sim G^{1,1}_{1,2}(-z)$, $\Psi(a,c;z)\sim G^{2,1}_{1,2}(z)$; the Bessel functions $J_\nu(z)\sim G^{1,0}_{0,2}(-z^2/4)$ and all their modifications as I_ν, as the McDonald function $K_\nu(z)\sim G^{2,0}_{0,2}(z^2/4)$, the functions of Lommel, Struve, etc.; all the classical orthogonal polynomials (and the associated special functions) as these of: Laguerre $L_n^{(\alpha)}$, Jacobi $P_n^{(\alpha,\beta)}$, Gegenbauer C_n^ν, Legendre P_n, Hermite H_n, etc; the functions of the parabolic cylinder, the incomplete gamma- and beta- and the error-functions, the Airy functions, etc; as well as the elementary functions like: $(1\pm z)^{-\alpha}\sim G^{1,0}_{1,1}(\mp z), |z|<1, \alpha<-1$; $\exp(z)=G^{1,0}_{0,1}(-z)$, $\ln(1+z)\sim G^{1,2}_{2,2}(z)$, $\sin(z)$, $\cos(z)\sim G^{1,0}_{0,2}(z^2/4)$, $\arcsin(z), \arctan(z)\sim G^{1,2}_{2,2}(z^2)$, etc.

Thus, instead of the tiresome task to learn a tremendous variety of necessary facts for all these separate classes of functions, one can take the advantage of any knowledge on the G-function only. Nowadays, from a complicated and fearful object to deal with, it becomes an effective and favourite tool for many mathematicians and applied scientists.

From the simple operational properties of the G-function, that can be found displayed on 1-2 pages only, in any of the mentioned handbooks, one can derive as particular cases, corresponding properties for each of the special functions, as some identities, recurrence and differential relations, series expansions, integrals involving them, etc. The asymptotic behaviour of the G-function, in neighbourhods of $z=0$ and $z=\infty$ has been studied yet by Meijer [22] and Braaksma [2], see in the handbooks, and near the third singular point $z = (-1)^{p-m-n}$ (when $p = q$), it has been clarified more recently, mainly due to Marichev, see [16] (reported and published in Bulgaria!), and [26].

The great role of the G-function is justified, besides by the above examples, by the fact that it satisfies the *generalized hypergeometric differential equation*

$$(4) \qquad \left[(-1)^{p-m-n}\, z \prod_{j=1}^{p}\left(z\frac{d}{dz} - a_j + 1\right) - \prod_{k=1}^{q}\left(z\frac{d}{dz} - b_k\right)\right] y(z)$$

$$= \left[(-1)^{p-m-n}\, P(z\frac{d}{dz}) - Q(z\frac{d}{dz})\right] y(z) = 0, \quad (P, Q - \text{polynomials of degrees } p, q)$$

of order $\max(p,q)$, having 2 singular points (if $p \neq q$): $z = 0$ (regular) and $z = \infty$ (irregular); or 3 regular singular ponts (if $p = q$): $z = 0, z = (-1)^{p-m-n}, z = \infty$. Thus, the G-function provides solutions to a very large class of singular ODEs in mathematical physics, of arbitrary order and with variable coefficients. Such type ODEs were, for example, these considered by Pochhammer and Goursat, that motivated the Italian mathematician Pincherle to introduce yet in 1888, as a pioneer, Mellin-Barnes type integrals like (2), see the historical comments in [15].

Although the G-function is rather general, yet there exist few examples of "exotic" special functions that in the general cases could not be presented in terms of (2), as for example: the Mittag-Leffler function $E_{\rho,\mu}(z)$, the Wright generalized hypergeometric function ${}_p\Psi_q\left((a_j, A_j)_1^p; (b_k, B_k)_1^q; z\right)$, the Bessel-Maitland (Wright) function $J_\nu^{(\mu)}(z)$, etc. This has made necessary the *next step* in extending the generalized hypergeometric functions, to the so-called *H-function of Fox*, see Fox [8] as well as handbooks like [26], [10]. These are same type Mellin-Barnes integrals but involving additional parameters $A_j > 0, B_k > 0$ in the arguments of the Γ-functions,

$$(5) \qquad H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_j, A_j)_1^p \\ (b_k, B_k)_1^q \end{matrix}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\prod_{k=1}^{m}\Gamma(b_k - B_k s)\prod_{j=1}^{n}\Gamma(1 - a_j + A_j s)}{\prod_{k=m+1}^{q}\Gamma(1 - b_k + B_k s)\prod_{j=n+1}^{p}\Gamma(a_j - A_j s)}\, z^s ds.$$

When $A_j = B_k = 1$, or all are positive rational numbers, (5) reduces to a G-function, see Boersma [1] (and same is for above mentioned special cases of (5)).

In my studies on title's topics, for many years I had the pleasant experience *to find the G-function* (and recently, its extension (5)) *essentially necessary and useful, and closely related to some traditionally Bulgarian trends.*

2. Relation to Obrechkoff transform and hyper-Bessel operators and equations. For the first time, I have "discovered" the G-function of Meijer in 1974, [9], as the "unknown" special function in the kernel of the Obrechkoff integral transform. Later, my studies on this transform and on the related "Bessel type" operators have become an endless source for further G-function's applications, and completely new achievements.

The Obrechkoff transform is one of the most general Laplace- and Meijer-type integral transforms and is related to the Bessel-type differential operators of arbitrary order $m \geq 1$, introduced by Dimovski [4], known recently as *hyper-Bessel differential operators* and represented in the equivalent forms ($0 < z < \infty$):

$$B = z^{-\beta} P(z\frac{d}{dz}) = z^{-\beta} \prod_{k=1}^{m} \left(z\frac{d}{dz} + \beta\gamma_k \right) = z^{\alpha_0} \frac{d}{dz} z^{\alpha_1} \frac{d}{dz} \ \ldots \ z^{\alpha_{m-1}} \frac{d}{dz} z^{\alpha_m}$$

$$= z^{-\beta} \left(z^m \frac{d^m}{dz^m} + a_1 z^{m-1} \frac{d^{m-1}}{dz^{m-1}} + \cdots + a_{m-1} z \frac{d}{dz} + a_m \right), \tag{6}$$

with some $\beta > 0$ and real $\gamma_k, k = 1, \ldots, m$, or with α_j's and a_k's determined and restricted suitably. Operators (6) include the usual differentiation $D_m = \frac{d^m}{dz^m}$ of order m ($\forall \alpha_j = 0, \beta = m$), the 2nd order Bessel differential operator B_ν related to the Bessel differential equation for $J_\nu(z)$ ($m = 2, \gamma_{1,2} = \pm\nu/2$) and many others, appearing often in differential equations of mathematical physics (in hydro-aerodynamics, axially/biaxially symmetric potential problems, elasticity, generalized heat transfer, etc), especially when using polar or cylinrical coordinates. Dimovski [4] has developed an operational calculus for the operators (6), following the algebraic method of Mikusinski and based on the notion "convolution" of the linear right inverse operator L to B. This operator, $y(z) = Lf(z)$ is defined as the solution of a Cauchy problem for $By(z) = f(z)$ under so-called "zero hyper-Bessel initial conditions" and the name *"hyper-Bessel integral operator"* has been adopted for it, see [10].

As an integral transformation, suitable for building alternative operational calculus for B, L, and generalizing the Laplace transform for the classical operational calculus, Dimovski [5] has considered the *Obrechkoff transform*

$$\mathcal{O}\{f; z\} = \beta \int_0^\infty z^{\beta(\gamma_m+1)-1} \ K\left[(zt)^\beta\right] f(t)dt \tag{7}$$

with the kernel-function

$$K(z) = \int_0^\infty \ldots \int_0^\infty \exp\left(-u_1 - \ldots - u_{m-1} - \frac{z}{u_1 \ldots u_{m-1}} \right) \prod_{k=1}^{m} u_k^{\gamma_m - \gamma_k - 1} du_1 \ldots du_{m-1}. \tag{8}$$

As a matter of fact, this is a slight modification of the integral transform introduced by Obrechkoff (1958, [25]) for other purposes (to generalize the famous Bernstein's theorem on the integral representation of absolutely monotonic functions). For the same goal, in an early paper [24], Obrechkoff has studied a simpler transformation, almost coinciding with the *integral transform of Meijer*, [20] (special case of (7) for $m = 2$). Dimovski [5] has shown that the Obrechkoff transformation (7) algebrizes the hyper-Bessel differential and integral operators B, L and has found a family of convolutions for $\mathcal{O}$ which are also convolutions of the operator L, thus justifying its use as a transform basis of the "hyper-Bessel operational calculus".

Essential for the further studies on the Obrechkoff transform and simplification of all its theory, has been the *identification of the kernel-function (8) as a Meijer's* $G^{m,0}_{0,m}$-function (see Kiryakova [9], [10], Dimovski and Kiryakova [6]), thus giving a new definition for the $\mathcal{O}$-transfomation (7):

$$\mathcal{O}\{f; z\} = \beta z^{-\beta(\gamma_m+1)+1} \int_0^\infty G^{m,0}_{0,m} \left[(zt)^\beta \,\middle|\, \begin{matrix} - \\ (\gamma_k - \frac{1}{\beta} + 1)_1^m \end{matrix} \right] f(t)dt. \tag{9}$$

Then, this is a special case of the so-called *G-transforms*, studied intensively in the recent years by many authors (see Vu Kim Tuan [28], Kiryakova [10, Ch.5]),

$$\mathcal{G}(z) = \{\mathcal{G}f(t); z\} = \int\limits_0^\infty G^{m,n}_{p,q}\left[zt\,\middle|\, \begin{matrix}(\alpha_j)\\(\beta_k)\end{matrix}\right] f(t)dt. \tag{10}$$

Based on (9), in a series of papers, we have found a full list of operational properties for the Obrechkoff transform, complex and real inversion formulas, Abel-type theorems, etc. and its theory has been further developed in relation to the special functions and fractional calculus (see for example, Kiryakova [10, Ch.3], Dimovski and Kiryakova, in [29]). Next, an H-function analogue of the Obrechkoff transform has been studied (Musallam, Kiryakova, Vu [23]), related to more general differential and integral operators of fractional multiorder.

The hyper-Bessel integral operators themselves, happen to be another kind of G-transforms (10), using a $G^{m,0}_{m,m}$-function as a kernel, namely

$$y(z) = Lf(z) = \frac{z^\beta}{\beta^m}\int\limits_0^1 G^{m,0}_{m,m}\left[\sigma \,\middle|\, \begin{matrix}(\gamma_k+1)_1^m\\(\gamma_k)_1^m\end{matrix}\right] f\left(z\sigma^{1/\beta}\right)d\sigma, \tag{11}$$

and their fractional powers $L^\lambda, \lambda > 0$ have the same type representation but with $(\gamma_k + 1)_1^m$ replaced by $(\gamma_k+\lambda)_1^m$ in the upper row parameters of the G-function. These integral representations of L and L^λ have given rise to the generalized fractional calculus (see next section), in which B and L appear as generalized "fractional" derivatives and integrals of integer multiorder $(1, 1, \ldots, 1)$, and L^λ - of multiorder $(\lambda, \lambda, \ldots, \lambda)$.

Meanwhile, formula (11) involving the G-function, provides the solution of a Cauchy problem for the differential equation $By(z) = f(z)$. To solve a nonhomogeneous hyper-Bessel differential equation as $By(z) = \lambda y(z) + f(z)$ under arbitrary (nonzero) initial conditions, we have derived the following result. Namely, if $\gamma_1 < \gamma_2 < \cdots < \gamma_m$, the fundamental system of solutions of the equation $By(z) = \lambda y(z)$, in a neighbourhood of $z = 0$, consists of Meijer's G-functions:

$$y_k(z) = G^{1,0}_{0,m}\left[-\lambda\frac{z^\beta}{\beta^m}\,\middle|\, \begin{matrix}-\\-\gamma_k, -\gamma_1, \ldots, -\gamma_{k-1}, -\gamma_{k+1}, \ldots, -\gamma_m\end{matrix}\right], \tag{12}$$

$k = 1, \ldots, m$, since this equation falls as a very special case of the generalized hypergeometric differential equation (4) with $p = 0, q = m$. Up to constant multipliers, (12) are "Bessel-type" generalized hypergeometric functions ${}_0F_{m-1}$ (see Section 4), or can be equivalently expressed by means of the *hyper-Bessel functions of Delerue* (multiindex analogues of the Bessel function J_ν, corresponding to hyper-Bessel operators):

$$J^{(m-1)}_{\nu_1,\ldots,\nu_{m-1}}(z) = \frac{(z/m)^{\nu_1+\cdots+\nu_{m-1}}}{\Gamma(\nu_1+1)\ldots\Gamma(\nu_{m-1}+1)}\,{}_0F_{m-1}\left((\nu_k+1)_1^{m-1}; -(z/m)^m\right). \tag{13}$$

In other series of works, we have shown that the G-functions are essentially involved also in the solutions to hyper-Bessel integral equations of Volterra type, as well as to pairs of dual integral equations, see details in [10], [13], etc.

3. Generalized fractional calculus. Another special class of the G-transforms (10) appearing in my studies, are the so-called *operators of generalized fractional calculus ("GFC")*, involving Meijer's G-functions as kernels.

The notion *"generalized operators of fractional integration"* has been recently adopted for the linear singilar integral operators of the form

$$If(z) = z^{\delta_0} \int\limits_0^1 \Phi(\sigma)\sigma^{\gamma} f(z\sigma)d\sigma, \quad \delta_0 \geq 0, \gamma \in \mathbf{R},$$

with an arbitrary elementary or special function Φ as a kernel, whose operational properties have to simulate the basic rules of the classical (Riemann-Liouville, R-L) fractional calculus, [27]. However, taking the kernel-functions to be arbitrary G- or H-functions, leads to unnecessary extent of generalization, allowing only formal considerations. The hint how to restrict suitably the class of Meijer's G-functions for kernels Φ, allowing to develop a meaningfull and detailed theory of generalized fractional calculus with practical applications (Kiryakova [10]), has come from the representations (11) of the hyper-Bessel integral operators L, L^{λ}.

The *generalized fractional integrals* of multiorder $\delta = (\delta_1 \geq 0, \dots, \delta_m \geq 0)$, basic for our GFC, are compositions of $m > 1$ commuting Erdélyi-Kober (E-K) fractional integrals, [27], [10] $(k = 1, \dots, m)$

$$(14) \qquad I_{\beta}^{\gamma_k,\delta_k} f(z) = \int\limits_0^1 \frac{(1-\sigma)^{\delta_k - 1}\sigma^{\gamma_k}}{\Gamma(\delta_k)} f(z\sigma^{\frac{1}{\beta}})\, d\sigma, \quad \delta_k > 0, \gamma_k \in \mathbf{R}, \beta > 0,$$

but defined explicitly by *single integrals involving Meijer's G-functions*:

$$(15) \qquad I_{\beta,m}^{(\gamma_k),(\delta_k)} f(z) := \int\limits_0^1 G_{m,m}^{m,0}\left[\sigma \,\middle|\, \begin{matrix} (\gamma_k + \delta_k)_1^m \\ (\gamma_k)_1^m \end{matrix}\right] f\left(z\sigma^{1/\beta}\right) d\sigma, \quad \text{if} \quad \sum_{k=1}^{m} \delta_k > 0.$$

The respective *generalized fractional derivatives* $D_{\beta,m}^{(\gamma_k),(\delta_k)}$ have explicit integro-differential representations [10], coming from the idea of the classical R-L derivatives [27].

The case $m = 1$ gives the E-K operators (14), and in particular the Riemann-Liouville fractional integrals $R^{\delta} = z^{\delta} I_1^{0,\delta}$, as well as all their special cases widely used in analysis; for $m = 2$ the $G_{2,2}^{2,0}$-function reduces to a Gauss hypergeometric function and from (15) we get the hypergeometric fractional integrals; some examples of operators (15) for $m = 3$ and including Horn's F_3-functions have been studied by Marichev, Saigo, etc. For arbitrary $m > 1$ we have from the operators of GFC, the hyper-Bessel operators: $B = D_{\beta,m}^{(\gamma_k),(1,\dots,1)} z^{-\beta}$, $L = z^{\beta} I_{\beta,m}^{(\gamma_k),(1,\dots,1)}$. Many other generalized differentiation and integration operators are special cases in the scheme of GFC, some of them - in univalent function theory, others - as transmutation operators for solving differential and integral equations, in operational calculus, or in representation formulas for special functions.

We omit the details of the corresponding functional spaces (of weighted continuous, integrable or analytic functions) and appropriate conditions on parameters. Just a short list of some *rules of the generalized fractional calculus* gives an idea how useful for their derivation the properties of the kernel G-function could be:

$$I_{\beta,m}^{(\gamma_k),(\delta_k)}\{z^p\} = c_p z^p, \quad c_p = \prod_{k=1}^{m} [\Gamma(\gamma_k + 1 + p/\beta)/\Gamma(\gamma_k + \delta_k + 1 + p/\beta)];$$

$$I_{\beta,m}^{(\gamma_k),(\delta_k)} z^{\lambda} f(z) = z^{\lambda} I_{\beta,m}^{(\gamma_k + \lambda/\beta),(\delta_k)} f(z) \quad \text{(commutability with power functions);}$$

$$I_{\beta,m}^{(\gamma_k),(\delta_k)} I_{\beta,m}^{(\tau_j),(\alpha_j)} f(z) = I_{\beta,m}^{(\tau_j),(\alpha_j)} I_{\beta,m}^{(\gamma_k),(\delta_k)} f(z) = I_{\beta,m+n}^{(\gamma_k,\tau_j),(\delta_k,\alpha_j)} f(z)$$

(commutability and compositions of operators (18));

$$I_{\beta,m}^{(\gamma_k+\delta_k),(\sigma_k)} I_{\beta,m}^{(\gamma_k),(\delta_k)} f(z) = I_{\beta,m}^{(\gamma_k),(\sigma_k+\delta_k)} f(z) \quad \text{(product rule, semigroup property)};$$

$$\left\{ I_{\beta,m}^{(\gamma_k),(\delta_k)} \right\}^{-1} f(z) = I_{\beta,m}^{(\gamma_k+\delta_k),(-\delta_k)} f(z) := D_{\beta,m}^{(\gamma_k),(\delta_k)} f(z); \quad \text{(inversion formula)}.$$

Our next generalizations have concerned fractional calculus operators and integral transforms of Laplace type involving Fox's H-functions, see [12], [23].

4. Applications to special functions. Usually, the special functions of mathematical physics, being special cases of the ${}_pF_q$*-functions* (compare (3)):

$$(16) \qquad {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{z^k}{k!}, \quad (a)_k := \frac{\Gamma(a+k)}{\Gamma(a)},$$

$|z| < \infty$ if $p \leq q$, and $|z| < 1$ if $p = q + 1$, are defined by means of their power series representations, like (16). However, some alternative representations can be used as their definitions, as: the well-known *Poisson integrals* for the Bessel functions and the analytical continuation of the Gauss hypergeometric function via the *Euler integral formula*. Also, some special functions have representations by *repeated*, or even *fractional differentiation*. Such examples are given by the *Rodrigues differential formulas* for the clasical orthogonal polynomials, used often as their basic definitions. There exists a variety of other integral and differential formulas, quite peculiar for each corresponding special function and scattered in the literature without any common idea.

In [10], [11] we have proposed a *unified approach in deriving all such formulas and their generalizations* (new or newly written integral, differential and differintegral representations of the special functions) by means of the generalized fractional calculus. A suitable classification of these special functions is also introduced. *The idea is based on the following simple facts*: i) most of the special functions of mathematical physics are nothing but modifications of the *generalized hypergeometric functions (g.h. f-s)* ${}_pF_q$; ii) each ${}_pF_q$-function can be represented as an E-K fractional differintegral of a ${}_{p-1}F_{q-1}$-function; iii) a finite number (q) of steps ii) leads to one of the basic g.h.f-s: ${}_0F_{q-p}$ (for $q-p = 1$: Bessel function), ${}_1F_1$ (confluent h.f.), ${}_2F_1$ (Gauss h.f.); iv) these 3 basic g.h.f-s can be considered as fractional differintegrals of 3 elementary functions, depending on whether $p < q, p = q$ or $p = q + 1$, namely:

$$(17) \qquad \cos_{q-p+1}(z) \ , \quad z^{\alpha} \exp z \ , \quad z^{\alpha}(1-z)^{\beta};$$

v) the compositions of E-K operators arising in iii) give generalized (q-tuple) fractional integrals (15), or corresponding derivatives. Thus, we obtain the following basic proposition:

All the generalized hypergeometric functions ${}_pF_q$ *can be considered as generalized (q-tuple) fractional differintegrals* $I_{\beta,q}^{(\gamma_k),(\delta_k)}$, $D_{\beta,q}^{(\gamma_k),(\delta_k)}$ *of one of the elementary functions (17), depending on if* $p < q \, , \, p = q \, , \, p = q + 1$.

In the above denotation, $y(z) = \cos_m(z)$ stands for the so-called *generalized cosine function*, the solution of the Cauchy problem

$$y^{(m)}(z) = -y(z), \quad y(0) = 1, y^{(k)}(0) = 0, \, k = 1, \ldots, m-1, \quad \text{namely:}$$

(18) $$\cos_m(z) := {}_0F_{m-1}\left(\left(\frac{k}{m}\right)_1^{m-1}; -\left(\frac{z}{m}\right)^m\right) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{mk}}{(mk)!}.$$

Practically, this statement is described in details by several lemmas and theorems, that could not be shortly summarized here, see the details in [10], [11]. They give us new integral and differential formulas for the special functions, some of them extending the Poisson or Euler integral formulas, others - the Rodrigues differential formulas; and even some numerical algorithms for their calculation. Each of them proposes a new sight on the generalized hypergeometric functions ${}_pF_q$ of the *three classes* formed in such a way, and called respectively: 1) $p < q$: *g.h.f-s of Bessel type* (the $\cos_m$-function, the Bessel function, the hyper-Bessel function (13), etc.); 2) $p = q$: *g.h.f-s of confluent type* (e.g. the exp-function, Kummer confluent function ${}_1F_1$, Laguerre polynomials, error function, etc.); 3) $p = q + 1$: *g.h.f-s of Gauss type* (the Gauss function ${}_2F_1$, the Jacobi, Legendre etc. polynomials). Such a scheme proposes, especially to applied scientists and engineers, a more comprehensive view on the complicated special functions – like GFC-images of the simplest functions (17), with similar to their properties (at least in asymptotical or pedagogical sense). Studying of the special functions can be simplified, by their "reduction" via the generalized fractional integrals and derivatives involving G-function as a kernel, to few simpler functions: elementary ones like (17), or much better known special functions, like the Bessel, Kummer and Gauss functions.

From the short discussions in this section, one can be again convinced in the peculiar role that Meijer's G-function plays, even *among the other special functions from which it has emerged*!

5. Comments. Discussing the subject of this survey, one can make the following parallel. Meijer has published the results on his main contributions (integral transform and G-function bearing his name) only in Dutch journals, [19]-[22]. This made the G-function unknown to wider mathematical community for some decade, or more (until [7] appeared). The famous Bulgarian mathematician Obrechkoff evidently was not familiar with these Meijer's works. On his side, Obrechkoff never tried to publish his results from [24], [25] (in Bulgarian language) in some international journals. Thus, transformation (7) was rediscovered many times in rather special cases by foreign mathematicians looking for Laplace type transforms for operational calculi related to Bessel-type operators. His achievements and priority became popular only after Dimovski's publications, like [5].

The Bulgarian group working in directions as Sections 2),3),4) have stimulated and emphasized the G-function's applications and generalizations of Meijer's transform, also by meetings and publications as [29], [30].

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G-ФУНКЦИЯТА НА МАЙЕР: БЪЛГАРСКАТА СЛЕДА ЗА НЕЙНАТА УПОТРЕБА В СПЕЦИАЛНИТЕ ФУНКЦИИ, ИНТЕГРАЛНИТЕ ТРАНСФОРМАЦИИ И ДРОБНОТО СМЯТАНЕ

Виржиния С. Кирякова

Тази работа представя кратък обзор за ролята на G-функцията на Майер в развитието на указаните направления, свързани с работите на някои български математици, и по-специално в изследванията на автора от 1974 г. насам. Той е инспириран от скорошна визита в Университета в Грьонинген, Холандия, мястото където проф. Майер е учил и работил, и където сега бе отбелязана и българската следа в това типично за холандската математика направление. Друга добра провокация е нарастналото внимание, което напоследък чуждестранни автори обръщат върху историята, развитието и актуалните приложения на т. нар. интеграли от типа на Мелин-Бърнс, от дефиницията на G-функцията. Оказва се, че тази обобщена хипергеометрична функция предлага решения на широк клас сингулярни диференциални уравнения от произволен ред с променливи коефициенти, както и на интегрални уравнения от типа на Волтера; също ядра на множество от линейни интегрални и интегро-диференциални оператори в приложния анализ; както и нова схема за унификация на специалните функции.