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BOUNDS FOR TERNARY EQUIDISTANT CONSTANT WEIGHT CODES

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In this paper we explore the problem of finding bounds for equidistant constant weight codes with $2 \leq w < n \leq 10$. Optimal ternary equidistant constant weight codes have been constructed by combinatorial and computer methods.

1. Introduction. Consider a finite set of q elements and containing a distinguished element "zero". The choice of a set does not matter in our context and we will use the set Z_q of integers modulo q. Let Z_q^n be the set of n-tuples (or vectors) over Z_q and $Z_q^{n,w}$ be the set of n-tuples over Z_q of Hamming weight w.

A code is called *constant weight* if all the codewords have the same weight w. A code is called *equidistant* if all the distances between distinct codewords are d. Let $B_q(n,d)$ denote the maximum number of codewords in an equidistant code over Z_q of length n and distance d (called an $(n,M,d;q)$ equidistant code) and $B_q(n,d,w)$ denote the maximum number of codewords in an equidistant constant weight code over Z_q of length n, distance d, and weight w (called an $(n,M,d,w;q)$ equidistant constant weight code ECWC).

Equidistant codes have been investigated by a large number of authors, mainly as examples of designs and other combinatorial objects. Some works published on this topic are [5], [6], [8], [12]. Constant weight codes have been studied by many authors. For some references for the binary case, see Brouwer *et al.* [3], Agrell [1] and for the ternary case, see Bogdanova [2] and Svanström [11]. A few papers study codes which are both equidistant and of constant weight, for example [7], [10] and [4]. The same problem is considered in this paper.

2. Preliminaries. Some bounds for ECWC are given by the following theorems:

Theorem 1 (Trivial values)**.**

$$B_3(n,d,w) = 1 \ if \ d > 2w,$$

$$B_3(n,d,n) = 2.$$

Theorem 2 (the Johnson bounds for ECWC)**.** *The maximum number of codewords in a q-ary ECWC satisfy the inequalities:*

$$B_q(n,d,w) \leq \frac{n}{n-w} B_q(n-1,d,w),$$

$$B_q(n,d,w) \le \frac{n(q-1)}{w} B_q(n-1,d,w-1).$$

The proof of the Theorem 2 is the same as the proof of Johnson bound for constant-weight codes [11].

Theorem 3 [4]. *For* $k = 1,2,\ldots,n,$ *if* $P_k^2(w) > P_k(d)P_k(0)$*, then*

$$B_q(n,d,w) \le \frac{P_k^2(0) - P_k(d)P_k(0)}{P_k^2(w) - P_k(d)P_k(0)}.$$

Here $P_k(x)$ is the Krawtchouk polinomial defined by

$$P_k(x) = \sum_{i=0}^{k} \binom{x}{i}\binom{n-x}{k-i}(-1)^i(q-1)^{k-i}$$

and

$$P_k(0) = \binom{n}{k}(q-1)^k.$$

According to [7] there are no ECWC of order q, lenght $q+1$, distance q and weight $q-1$ which have more than $\frac{(q^2+q)}{2}$ codewords, regardless whether q is even or odd.

3. Some combinatorial bounds and constructions of ECWC.

Proposition 4. *There exists a family of optimal ternary ECWC with parameters* $(n,3,3,2;3)$ *for every integer* $n \ge 3$.

Proof. Let u be a fixed codeword with length n and weight 2. Consider how many codewords are at distance exactly 3 from u we obtain that $B_3(n,3,2) = 3$. ■

Proposition 5. *For* $w = 2,\ldots,n,$ $\quad P_n(w) \ne 0$.

Proof.

$$P_n(w) = \sum_{i=0}^{n} \binom{w}{i}\binom{n-w}{n-i}(-1)^i(q-1)^{n-i}.$$

For the validity of binomial coefficients the following conditions must be satisfied:

$$\left| \begin{array}{l} i \le w \\ n-i \le n-w \end{array} \right. .$$

Therefore $i = w$. Then $P_n(w) = (-1)^w(q-1)^{n-w} \ne 0$. ■

Proposition 6. *For* $d = 3,$ $w = 3,\ldots,n$ *and* $k = n,$

$$B_q(n,d,w) < (q-1)^3 + 1.$$

Proof.

$$B_q(n,d,w) \le \frac{P_k^2(0) - P_k(0)\,P_k(d)}{P_k^2(w) - P_k(0)\,P_k(d)}$$

For $d = 3$ and $k = n$ and using the same reasoning as in the Proposition 5 we obtain that

$$P_n(3) = \sum_{i=0}^{n} \binom{3}{i}\binom{n-3}{n-i}(-1)^i (q-1)^{n-i} = -(q-1)^{n-3}.$$

We have $P_n(0) = \binom{n}{n}(q-1)^n = (q-1)^n$ and consequently

$$B_q(n,d,w) \le \frac{(q-1)^{2n} + (q-1)^{2n-3}}{P_k^2(w) + (q-1)^{2n-3}} < (q-1)^3 + 1.$$

■

Corollary 7. *There exists a family of optimal ternary ECWC with parameters* $(4+\lambda, 8, 3, 3+t; 3)$ *for every integer* $\lambda \ge 0$ *and* $0 \le t \le n-3$.

Proof. From the Simplex code S, which has parameters $(4, 9, 3; 3)$ we construct ECWC $C = S \setminus \{0\}$. From the code C we construct a family of $(4+\lambda, 8, 3, 3+t; 3)$ ECWC C' in the following way:

$$C' = \left\{ \left(\underbrace{00\ldots0}_{\lambda-t}\underbrace{11\ldots1}_{t}, c \right) | c \in C \right\},$$

where $\lambda \ge 0$ and $0 \le t \le n-3$. Therefore $B_3(n,3,w) \ge 8$. For these parameters Proposition 6 gives $B_3(n,3,w) < 9$. So

$$B_3(n,3,w) = 8.$$

■

4. Bounds for ECWC. For codes of small size we apply combinatorial reasoning. For the rest of the values of M we use specificly developed, computer algorithms.

The best known upper and lower bounds (and exact values when these coincide) for ternary ECWC of lenght $n \le 10$ are displayed in Table 1. If in a certain position only one number occurs, then this number is the exact value of $B_3(n,d,w)$ and the corresponding codes are optimal. If two numbers are given, then the right one is the best known upper bound for $B_3(n,d,w)$, received from Theorem 2 and Theorem 3. The left one is the best known lower bound for $B_3(n,d,w)$, received by our computer algorithm (exhaustive search), which is of exponential complexity.

All the numbers in column $d = 3$ are obtained by Corollary 7.

Table 1. Bounds for optimal ternary ECWC

n	w	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$	$d=9$	$d=10$
3	2	3							
4	2	3	2						
	3	8	2						
5	2	3	2						
	3	8	5	2					
	4	8	5	2					
6	2	3	3						
	3	8	5	4	2				
	4	8	6	4	3				
	5	8	6	3	2				
7	2	3	3						
	3	8	7	4	2				
	4	8	7	7	3	2			
	5	8	6	6	3	2			
	6	8	6	7	2	2			
8	2	3	4						
	3	8	7	4	2				
	4	8	7	7	5	2	2		
	5	8	8	7	8	3	2		
	6	8	6	7	8	2	2		
	7	8	8	8	4	2	2		
9	2	3	4						
	3	8	7	4	3				
	4	8	7	7	9	3	2		
	5	8	8	7	9	5	3	2	
	6	8	8	7	11	6	3	3	
	7	8	8	8	12	5	3	2	
	8	8	8	8	9	3	2	2	
10	2	3	5						
	3	8	7	4	3				
	4	8	7	7	15	5	2		
	5	8	8	7	12 - 21	8 - 9	4	2	2
	6	8	8	7	14 - 20	8 - 15	5	3	2
	7	8	8	8	11 - 20	9 - 16	5	3	2
	8	8	8	8	12 - 21	10 - 11	5	2	2
	9	8	8	8	10	5	2	2	2

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ГРАНИЦИ ЗА ТРОИЧНИ ЕКВИДИСТАНТНО КОНСТАНТНО ТЕГЛОВНИ КОДОВЕ

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Изследван е проблема за намиране на граници за троични еквидистантни константно тегловни кодове при $2 \leq w < n \leq 10$. Използвани са комбинаторни и компютърни методи за конструиране на оптимални троични еквидистантни константно тегловни кодове.