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BISECTION METHOD FOR SOLVING A GAME WITH LOCALLY LIPSCHITZ PAYOFF

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In this paper, it is suggested an iterative method for finding out the saddle point of an antagonistic game over the unit square with locally Lipschitz payoff, which is pseudoconcave with respect to the first argument and pseudoconvex with respect to the second argument.

1. Introduction. Let S be a nonempty open set in the finite-dimensional Euclidean space $\mathbb{R}^n$, the real-valued function f be defined on S, and $X \subset S$. Suppose that f is locally Lipschitz (l.L. for short). Denote its upper and lower Clarke generalized directional derivatives at the point x in the direction v by $f^{\uparrow}_{Cl}(x,v)$ and $f^{\downarrow}_{Cl}(x,v)$ respectively. Since the function $v \mapsto f^{\uparrow}_{Cl}(x,v)$ ($v \mapsto f^{\downarrow}_{Cl}(x,v)$) is sublinear (superlinear), then

$$f^{\uparrow}_{Cl}(x,v) = \max_{\xi\in\underline{\partial} f^{\uparrow}_{Cl}(x)} \langle \xi, v\rangle \ (f^{\downarrow}_{Cl}(x,v) = \min_{\xi\in\overline{\partial} f^{\downarrow}_{Cl}(x)} \langle \xi, v\rangle),$$

where $\underline{\partial} f^{\uparrow}_{Cl}(x)$ ($\overline{\partial} f^{\downarrow}_{Cl}(x)$) is the subdifferential (superdifferential) of this function. It is well-known that the sets $\underline{\partial} f^{\uparrow}_{Cl}(x)$ and $\overline{\partial} f^{\downarrow}_{Cl}(x)$ coincide [3, p. 73], and these convex compact sets are called the Clarke subdifferential of f at x. We shall denote it by $\partial f(x)$. An element of the Clarke subdifferential is called generalized gradient of the function f at the point x.

The original definition of pseudoconvexity by Mangasarian [6] in the differentiable setting was extended by many authors for nondifferentiable functions. For instance, see [9]: A l.L. function $f : X \to \mathbb{R}$ is said to be *pseudoconvex* (*strictly pseudoconvex*) on X if

$$\forall x, y \in X : \ f(y) < f(x) \ (x \neq y, \ f(y) \leq f(x)) \implies f^{\uparrow}_{Cl}(x, y-x) < 0. \tag{1}$$

The function f is called (strictly) pseudoconcave if $-f$ is (strictly) pseudoconvex on X, that is

$$\forall x, y \in X : \ f(y) > f(x) \ (x \neq y, \ f(y) \geq f(x)) \implies f^{\downarrow}_{Cl}(x, y-x) > 0.$$

We shall use the following well-known assertions.

Lemma 1. *If $f : X \to \mathbb{R}$ is a l.L. pseudoconvex function, then the implication (1) is equivalent to the following:*

$$\langle \xi, y-x\rangle < 0 \ \textit{for all}\ \xi \in \partial f(x).$$

Here by $\langle .,.\rangle$ we have denoted the usual scalar product in a finite-dimensional space.

Lemma 2. *If the l.L. function $f\colon X \to R$ is pseudoconvex, and $x^* \in X$, then $0 \in \partial f(x^*)$ implies that x^* is a global minimazer of $f(x)$ on the set X.*

Before stating the next assertion, we shall give another definition. The point $x_\epsilon \in X$ is said ϵ-minimal (ϵ-maximal) of $f(x)$ on X ($\epsilon \geq 0$) if $f(x_\epsilon) \leq f^* + \epsilon$ ($f(x_\epsilon) \geq f^* + \epsilon$), where f^* is the optimal value $f^* = \inf_{x \in X} f(x)$ ($f^* = \sup_{x \in X} f(x)$).

Lemma 3. *If X is a convex set, then the set X^*_ϵ of all ϵ-minimal (ϵ-maximal) points of the l.L. pseudoconvex (pseudoconcave) function $f(x)$, is convex.*

The following bisection method is a trivial generalization of the bisection method for differentiable pseudoconvex functions [1].

Let $[a, b]$ be a closed interval on the real line, and $f\colon X \to R$ be a l.L. pseudoconvex function. We search for the global minima of $f(x)$ on $[a, b]$. This minimizer exists, since $f(x)$ is continuous. Assume that we can find the subdifferential $\partial f(x)$ of $f(x)$ for all $x \in [a, b]$. Suppose that $[a, b]$ is the initial interval of undetermination of the global minimizer. Let λ be the middle of this interval. If $0 \in \partial f(\lambda)$, then by Lemma 2, λ is the global minimizer. Since $\partial f(\lambda)$ is a nonempty weak* compact set [2], then it is a closed interval. If $0 \notin \partial f(\lambda)$, then $\partial f(\lambda) \subset (0, \infty)$ or $\partial f(\lambda) \subset (-\infty, 0)$. In the first case $\xi(x - \lambda) \geq 0$ for all $x \in [a, b]$, such that $x \geq \lambda$, and for all $\xi \in \partial f(\lambda)$. According to Lemma 1 $f(x) \geq f(\lambda)$, whenever $x \geq \lambda$, and we may reduce the interval of indetermination to $[a, \lambda]$. In the second case we may reduce the interval of indetermination to $[\lambda, b]$. We repeat these operations as long as we get the necessary accuracy.

2. Main result. Consider the antagonistic game $\Gamma = \langle X,\ Y,\ f\rangle$. Here X and Y are the sets of strategies of the first and second players respectively, and $f(x, y)$ is the payoff of the first player. We suppose that X and Y are closed intervals on the real line, $f(\cdot, y)$ are l.L. pseudoconcave on X for all fixed $y \in Y$, $f(x, \cdot)$ are l.L. pseudoconvex on Y for all fixed $x \in X$.

The point $(\hat{x}, \hat{y}) \in X \times Y$ is said to be ϵ-saddle ($\epsilon \geq 0$) of the game Γ, see for example [12], if

$$f(x, \hat{y}) - \epsilon \leq f(\hat{x}, \hat{y}) \leq f(\hat{x}, y) + \epsilon \text{ for all } x \in X, y \in Y.$$

Strategies $\hat{x}$ and $\hat{y}$ are called ϵ-optimal.

We search for the saddle point of the game Γ. Denote by $\partial_x f(x, y)$ and $\partial_y f(x, y)$ the private subdifferentials of f at (x, y) with respect to x and y respectively. Suppose that we can calculate these subdifferentials everywhere on the Cartesian product $X \times Y$.

Assume that the game has a saddle point, and the pair $(x', y') \in X \times Y$ is such that x' is ϵ-optimal with respect to the maximum of the function $f(\cdot, y')$ on X, that is

$$f(x, y') - \epsilon \leq f(x', y') \text{ for all } x \in X. \tag{2}$$

There are 3 possible cases:

$$0 \in \partial_y f(x', y'),\ \partial_y f(x', y') \subset (0, \infty),\ \text{and } \partial_y f(x', y') \subset (-\infty, 0).$$

In the second case $\xi(y - y') \geq 0$ for all $y \in Y$ such that $y \geq y'$, and for all $\xi \in \partial_y f(x', y')$. Since $f(x', \cdot)$ is pseudoconvex, then $f(x', y) \geq f(x', y')$. It follows from (2) that for all $x \in X$, and for all $y \in Y$ such that $y \geq y'$

$$f(x, y') - \epsilon \leq f(x', y') \leq f(x', y) \tag{3}$$

Denote by $(\hat{x}, \hat{y})$ the searched saddle point. Then

$$f(x, \hat{y}) \leq f(\hat{x}, \hat{y}) \leq f(\hat{x}, y) \text{ for all } x \in X, y \in Y. \tag{4}$$

By (3) and (4), there exists a strategy $\tilde{y}$ of the second player such that $\tilde{y} \leq y'$ and $(\hat{x}, \tilde{y})$ is an ϵ-saddle point. Indeed, if $\hat{y} \leq y'$, then $\tilde{y} = \hat{y}$. Let $\hat{y} > y'$. If we take $x = \hat{x}$, $y = \hat{y}$

in (3), and $x = x'$, $y = y'$ in (4), then we shall get that $(\hat{x}, y')$ is an ϵ-saddle point, i.e. $\tilde{y} = y'$ is ϵ-optimal. We got that the interval $\{y \in Y \mid y \leq y'\}$ always contains a strategy, which composes with $\hat{x}$ an ϵ-saddle point. If $\partial_y f(x', y') \subset (-\infty, 0)$, then by analogy the interval $\{y \in Y \mid y \geq y'\}$ contains a strategy, which composes with $\hat{x}$ an ϵ-saddle point. If $0 \in \partial_y f(x', y')$, then $(\hat{x}, y')$ is an ϵ-saddle point, and y' is ϵ-optimal.

We shall use these arguments to show the convergence of the following algorithm. We choose the set Y for the initial interval of an undetermination of the ϵ-optimal strategy of the second player. Suppose for $k = 1,\ 2,\ 3, \dots$ that Y^k is the k-th interval of an undetermination, and y_k is the middle of this interval. Let the strategy of the first player x_k be ϵ-optimal with respect to the maximum of the function $f(\cdot, y_k)$ on X. There exist 3 possibilities:

$$0 \in \partial_y f(x_k, y_k),\ \partial_y f(x_k, y_k) \subset (0, \infty),\ \partial_y f(x_k, y_k) \subset (-\infty, 0).$$

In the first case y_k is an ϵ-optimal. In the second case the interval $\{y \in Y \mid y \leq y_k\}$ contains an ϵ-optimal strategy, which composes together with $\hat{x}$ an ϵ-saddle point. We shall prove that the interval $Y^{k+1} = \{y \in Y^k \mid y \leq y_k\}$ contains an ϵ-optimal strategy, which composes together with $\hat{x}$ an ϵ-saddle point. Otherwise, $Y^{k+1} \not\equiv \{y \in Y \mid y \leq y_k\}$, and there exist points

$$z_1 \in Y^k \setminus \{y \in Y \mid y \leq y_k\} \text{ and } z_2 \in \{y \in Y \mid y \leq y_k\} \setminus Y^k,$$

which are ϵ-optimal with respect to the global minimum of the function $f(\hat{x}, \cdot)$ on the set Y. According to Lemma 3 all points of the closed interval $[z_1, z_2]$ are ϵ-optimal. This conclusion contradicts the inclusion $Y^{k+1} \subset [z_1, z_2]$. Therefore Y^{k+1} contains an ϵ-optimal strategy. In the third case the interval $Y^{k+1} = \{y \in Y^k \mid y \geq y_k\}$ contains an ϵ-optimal strategy of the second player. We continue the decribed process until the lenght of the last interval of an undetermination is less than any preassigned positive number δ, however small.

By analogy we can get an ϵ-optimal strategy of the first player.

We shall use the following lemma.

Lemma 4 [12, Theorem 2.1.3]**.** *Consider the antagonistic game $\Gamma = \langle X, Y, f\rangle$ If the situations $(\hat{x}, \tilde{y})$ and $(\tilde{x}, \hat{y})$ are ϵ-saddle points ($\epsilon \geq 0$), then $(\tilde{x}, \tilde{y})$ is an 4ϵ-saddle point.*

As a consequence of Lemma 4 for all $k = 1,\ 2, \dots$ the rectangular $X^k \times Y^k$ contains an 4ϵ-saddle point for all sufficiently large k.

At last we show that the game has a saddle point. It is well-known that each l.L. pseudoconvex function is strictly quasiconvex. For example, this fact may be seen in the paper [8, Theorem 8], where pseudoconvex functions are called semiconvex. Since each strictly quasiconvex lower semicontinuous function, defined on convex set, is quasiconvex [5], then the functions $f(\cdot, y)$ are quasiconcave for all $y \in Y$, and the functions $f(x, \cdot)$ are quasiconvex for all $x \in X$. Thus, according to the following lemma due to Sion [10] the considered game has a saddle point.

Lemma 5. *Let X and Y be convex compact sets in $\mathbb{R}^n$. Suppose that the function $f(x, y)$ is defined on $X \times Y$. If the functions $f(x, \cdot)$ are lower semicontimuous and quasiconvex on Y for all $x \in X$, and the functions $f(\cdot, y)$ are upper semicontinuous and quasiconcave on X for all $y \in Y$, then the game $\Gamma = \langle X, Y, f\rangle$ has a saddle point.*

In our calculations we considered the Clarke generalized directional derivative, but instead of it can be used another convex derivatives as a function of the direction: the

Michel-Penot directional derivative [7] or asymptotic upper Dini derivative [11]. They have similar properties.

It is clear that this method may be used for solving maximin problems and matrix games with a large number of strategies.

If the functions $f(x,\cdot)$ are strictly pseudoconvex, and the functions $f(\cdot,y)$ are strictly pseudoconcave, then the saddle point is unique.

Considered algorithm were presented as abstract in [4].

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МЕТОД НА РАЗПОЛОВЯВАНЕТО ЗА РЕШАВАНЕ НА ИГРА С ЛОКАЛНО ЛИПШИЦОВА ПЛАТЕЖНА ФУНКЦИЯ

Всеволод Иванов Иванов

В статията е предложен итеративен метод эа намиране на седловата точка на антагонистична игра върху единичния квадрат с локално липшицова платежна функция, псевдовдлъбната по отношение на първия си аргумент и псевдоизпъкнала по отношение на втория.