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ON THE LINEAR COMBINATIONS OF SYMMETRIC SEGMENTS*

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We study linear combinations of symmetric line segments in the plane using the computer algebra system MATHEMATICA.
Key words: convex bodies, Minkowski operations, quasivector space.

1. Introduction. Convex bodies (and, in particular intervals) have been increasingly used in set-valued, convex and interval analysis. Therefore the algebraic operations on convex bodies, such as addition and multiplication by scalars, deserve special interest. Convex bodies form an abelian semigroup with cancellation law with respect to (vector, Minkowski) addition. Abelian semigroups with cancellation laws can be embedded into abelian groups. Clearly, it is more convenient to perform algebraic computations within a group than whithin a semigroup, where the elements are not invertible in general. This is the reason why the abelian group of (generalized, differences of, quotients of) convex bodies has been considered by a number of authors [2, 9, 10]. Multiplication by real scalar is usually defined in the additive group so that the group becomes a vector space [10]. However, this definition of multiplication by scalar is not an extension of the definition in the semigroup. An alternative definition has been proposed and studied in [5]–[8]; there a natural extension of the multiplication by real scalar is introduced, leading to a so-called quasivector space (q-linear space, quasilinear space with group structure).

It has been proved [8] that a quasivector space is a direct sum of a vector space and a symmetric quasivector space. Also a complete characterization of the symmetric quasivector spaces in the finite dimensional case is given. In practice this means that the symmetric case deserves special attention, cf. e. g. [3].

In this work we develop a MATHEMATICA program to calculate linear combinations of basic symmetric convex bodies in the sense of a quasivector space. As basic elements we consider symmetric segments, so that the linear combination is a zonotope. Here we restrict our considerations to the two-dimensional Euclidean plane $\mathbb{E}^2$.

2. Addition and Multiplication by Scalars. By $\mathbb{E}^n$, $n \geq 1$, we denote the n-dimensional real Euclidean vector space with origin 0. The ordered field of reals is denoted by $\mathbb{R}$. A convex compact subset of $\mathbb{E}^n$ is called *convex body (in* $\mathbb{E}^n$*)*; a convex body may not have necessarily interior points, e. g. a line segment and a single point in $\mathbb{E}^n$ are convex bodies [14]. The set of all convex bodies (in $\mathbb{E}^n$) will be denoted by

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$\mathcal{K} = \mathcal{K}(\mathbb{E}^n)$; $\mathbb{E}^n$ is a subset of $\mathcal{K}$, $\mathbb{E}^n \subset \mathcal{K}$. In the case $n = 1$ the elements of $\mathcal{K}(\mathbb{E}^n)$ are compact intervals on the real line.

Addition. The sum of two convex bodies $A, B \in \mathcal{K}$ (sometimes called vector or Minkowski sum [4]) is defined by

$$A + B \quad = \quad \{c \mid c = a + b,\ a \in A,\ b \in B\}, \quad A, B \in \mathcal{K}. \tag{1}$$

Clearly, (1) defines an algebraic operation in $\mathcal{K}$ called *addition* and $\mathcal{K}$ is closed under this operation. For $A, B, C \in \mathcal{K}$ we have, see e. g. [10]:

$$(A + B) + C \quad = \quad A + (B + C), \tag{2}$$

$$A + B \quad = \quad B + A, \tag{3}$$

$$A + 0 \quad = \quad A, \tag{4}$$

$$A + C = B + C \quad \Longrightarrow \quad A = B. \tag{5}$$

The equation $A + X = B$ may have a solution for certain pairs A, B. The solution X of $B + X = A$, if existing, is unique. Indeed, by definition, there is a $X \in \mathcal{K}$, such that $A = B + X$. Assume that $X' \in \mathcal{K}$, with $X' \neq X$ is such that $A = B + X'$. Then we have $B + X = B + X'$, which by the cancellation law (5) implies $X = X'$, a contradiction.

Multiplication by real scalars is defined by

$$\alpha * B \quad = \quad \{c \mid c = \alpha b,\ b \in B\}, \quad B \in \mathcal{K},\ \alpha \in \mathbb{R}. \tag{6}$$

Recall some properties of (6). For $A, B, C \in \mathcal{K}$, $\alpha, \beta, \gamma \in \mathbb{R}$, we have:

$$\gamma * (A + B) \quad = \quad \gamma * A + \gamma * B, \tag{7}$$

$$\alpha * (\beta * C) \quad = \quad (\alpha\beta) * C, \tag{8}$$

$$1 * A \quad = \quad A, \tag{9}$$

$$(\alpha + \beta) * C \quad = \quad \alpha * C + \beta * C, \quad \alpha\beta \geq 0. \tag{10}$$

3. Negation and symmetric elements. The operator neg: $\mathcal{K} \longrightarrow \mathcal{K}$ defined by $\operatorname{neg}(A) = (-1) * A = \{-a \mid a \in A\}$, $A \in \mathcal{K}$, is called *negation*, and will be symbolically denoted by $\neg A$.

For brevity, we denote for $A,\ B \in \mathcal{K}$

$$A \neg B = A + (\neg B) = \{a - b \mid a \in A,\ b \in B\}; \tag{11}$$

the operation $A \neg B$ is called *(outer) subtraction.*

We have $\neg(\gamma * A) = (-1) * (\gamma * A) = (-\gamma) * A = \gamma * (\neg A)$ for any real γ and $A \in \mathcal{K}$.

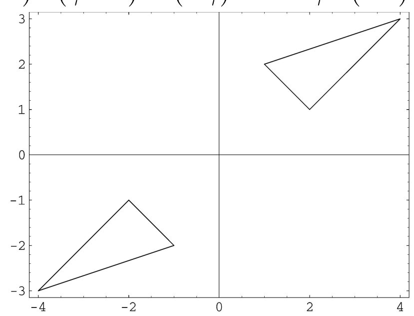

Figure 1: The operator negation

Symmetric bodies. An element $A \in \mathcal{K}$ is called *symmetric (with respect to the origin)*, if $x \in \mathbb{E}$, $x \in A$, implies $-x \in A$.

The set of all symmetric convex bodies is denoted by $\mathcal{K}_S$. We have $\mathcal{K}_S = \{A \in \mathcal{K} \mid A = \neg A\}$, i. e. $A \in \mathcal{K}$ is symmetric, if and only if $A = \neg A$. For $A \in \mathcal{K}$, the set $A \neg A$ is called the difference body of A (see [14, p. 127]). For $A \in \mathcal{K}$, we have $A \neg A \in \mathcal{K}_S$. Indeed, we have $\neg(A \neg A) = \neg A + A = A \neg A$.

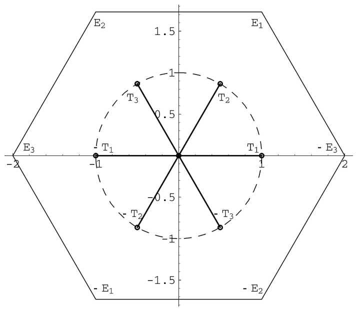

Figure 2: A linear combination of three segments — Example 1

The element $A \in \mathcal{K}$ is called *t-symmetric*, with *center* $t \in \mathbb{E}$, if $(A - t) \in \mathcal{K}_S$. In other words, a t-symmetric element is a t-translate of a symmetric element.

On Figure 2 the points T_1, T_2, T_3 and the corresponding symmetric points $-T_1, -T_2, -T_3$ define three symmetric line segments. The polygon (zonotope) with extreme points $E_1, E_2, E_3, -E_1, -E_2, -E_3$ is a symmetric convex body. We shall later discuss how the zonotope is generated by the symmetric line segments.

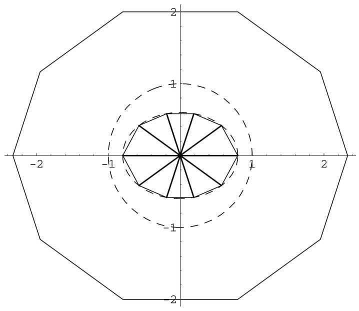

Figure 3: A linear combination of five segments — Example 2

4. Linear combinations and examples. The system $(\mathcal{K}, +)$ of all convex bodies of a real m-dimensional Euclidean vector space $\mathbb{E}^m$ with the operation addition (1) is an abelian monoid with cancellation law having as a neutral element the origin "0" of $\mathbb{E}^m$. The system $(\mathcal{K}, +, \mathbb{R}, *)$, where "$*$" is the operation multiplication by real scalars (6) is a quasilinear space (of monoid structure). The monoid $(\mathcal{K}, +)$ induces a group of generalized convex bodies $(\mathcal{D}(\mathcal{K}), +)$ cf. [2, 9, 10]. In [10] and some of the above cited literature the following multiplication by scalars has been used in $(\mathcal{D}(\mathcal{K}), +)$:

$$\gamma \cdot (A, B) = \begin{cases} (\gamma * A, \gamma * B), & \text{if } \gamma \geq 0, \\ (|\gamma| * B, |\gamma| * A), & \text{if } \gamma < 0. \end{cases} \tag{12}$$

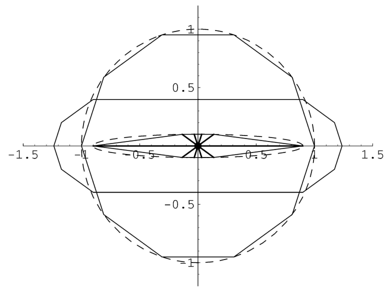

Figure 4: A linear combination of five segments — Example 3

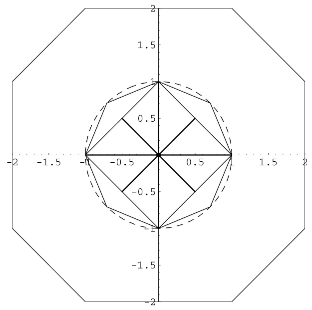

Figure 5: A linear combination of four segments — Example 4

As for $\gamma < 0$ we have $\gamma \cdot (A, 0) = (0, |\gamma| * A)$, which is an improper result, (12) is not an extension of the multiplication by scalars in $\mathcal{K}$ and seems to be of little practical value.

Multiplication by scalars "$*$" can be extended from $\mathbb{R} \times \mathcal{K}$ to $\mathbb{R} \times \mathcal{D}(\mathcal{K})$ by means of the following natural definition of $* : \mathbb{R} \times \mathcal{D}(\mathcal{K}) \longrightarrow \mathcal{D}(\mathcal{K})$ [8]:

$$\gamma * (A, B) = (\gamma * A, \gamma * B), \quad A,\ B \in \mathcal{K}, \quad \gamma \in \mathbb{R}. \tag{13}$$

In particular, multiplication by the scalar -1 in $\mathcal{D}(\mathcal{K})$, called *negation*, is denoted by $\neg(A, B) = (-1) * (A, B) = (\neg A, \neg B)$, $A,\ B \in \mathcal{M}$.

With the operation (13) the set of generalized convex bodies becomes a quasivector space in the following sense:

Definition 1. *A* quasivector space (over the l. o. field $\mathbb{R}$), *denoted* $(\mathcal{Q}, +, \mathbb{R}, *)$, *is an abelian group* $(\mathcal{Q}, +)$ *with a mapping (multiplication by scalars) "$*$": $\mathbb{R} \times \mathcal{Q} \longrightarrow \mathcal{Q}$, such that for $a, b, c \in \mathcal{Q}$, $\alpha, \beta, \gamma \in \mathbb{R}$:*

$$\gamma * (a + b) = \gamma * a + \gamma * b, \tag{14}$$

$$\alpha * (\beta * c) = (\alpha\beta) * c, \tag{15}$$

$$1 * a = a, \tag{16}$$

$$(\alpha + \beta) * c = \alpha * c + \beta * c, \quad \text{if } \alpha\beta \geq 0. \tag{17}$$

Definition 2. *Assume that Q is a quasivector space. The space $\mathcal{Q}' = \{a \in \mathcal{Q} \mid a \neg a = 0\}$ is called the* linear (distributive) subspace *of $\mathcal{Q}$ and the space $\mathcal{Q}'' = \{a \in \mathcal{Q} \mid a = \neg a\}$ is called the* symmetric (quasivector) subspace *of $\mathcal{Q}$.*

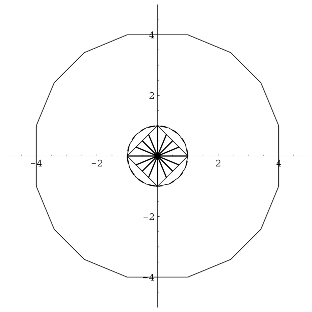

Figure 6: A linear combination of eight segments — Example 5

Theorem 1 [8]**.** *For every quasivector space $\mathcal{Q}$ we have $\mathcal{Q} = \mathcal{Q}' \oplus \mathcal{Q}''$. More specifically, for every $x \in \mathcal{Q}$ we have $x = x' + x''$ with unique $x' = (1/2) * (x + x_-) \in \mathcal{Q}'$, and $x'' = (1/2) * (x \neg x) \in \mathcal{Q}''$.*

Since a quasivector space is a direct sum of a vector space, and as vector spaces are well-known algebraic structures, we should concentrate on symmetric quasivector spaces. It has been shown [8] that in the latter spaces one can define linear combinations, linear dependency, basis, dimension, etc., as we do in a vector space. Following this theoretical background we next consider as a basis in the plane sets of symmetric line segments at equal angles.

Example 1. Consider three points T_1, T_2, T_3 on the unit circle and the corresponding symmetric points $-T_1, -T_2, -T_3$. The points define three symmetric line segments: $s_1 = (-T_1)T_1$, $s_2 = (-T_2)T_2$, $s_3 = (-T_3)T_3$. We take the points T_i so that the line segments form equal angles, see Figure 2. The polygon (zonotope) E with extreme points $E_1, E_2, E_3, -E_1, -E_2, -E_3$ is a symmetric convex body, which is the linear combination of the line segments s_i: $E = \sum s_i$ (in this simple case we take the coefficients in the linear combination to be one.

In the next examples we take the coefficients of the linear combination generally different from one, so that the reduced line segments lie on an ellipse, resp. on a diamond.

Example 2. In this example we take five segments with endpoints on the ellipse $x^2/a^2 + y^2/b^2 = 1$ with $a = 4/5, b = 3/5$, see Figure 3.

Example 3. In this example we take five segments with endpoints on the ellipse $x^2/a^2 + y^2/b^2 = 1$ with $a = 9/10, b = 1/10$, see Figure 4.

Example 4. In this example we take four segments with endpoints on a diamond defined by the lines $\pm x \pm y = 1$, see Figure 5.

Example 5. In this example we take eight segments with endpoints on a diamond defined by the lines $\pm x \pm y = 1$, see Figure 6.

For other similar examples see [1], where the problem of validated computation is considered.

REFERENCES

[1] R. Anguelov. Validated Computations with Convex Bodies, manuscript, 2002.

[2] R. Baier, E. Farkhi. Differences of Convex Compact Sets in the Space of Directed Sets. Part I: The Space of Directed Sets, Part II: Visualization of Directed Sets, *Set-Valued Analysis* **9** (2001) 217–245; 247–272.

[3] R. Faro Rivas. Approximation de cuerpos convexos simetricos. Thesis, Univ. de Extremadura, 1986, 147 pp.
[4] H. Hadwiger. Vorlesungen über Inhalt, Oberfläche und Isoperimetrie. Springer, Berlin, 1957.
[5] S. Markov. On the Algebra of Intervals and Convex Bodies, *J. UCS* **4** (1998) 34–47.
[6] S. Markov. On the Algebraic Properties of Convex Bodies, Pliska Stud. Math. Bulgar. 12 (1998), 119–132.
[7] S. Markov. On the Properties of Convex Bodies and Related Algebraic Systems, Ann. Univ. Sofia, Fac. Math. Inf., v. 91, livre 1, 41–59.
[8] S. Markov. On the Algebraic Properties of Convex Bodies and Some Applications, *J. of Convex Analysis* **7** (2000) 129–166.
[9] A. G. Pinsker. The Space of All Convex Sets of a Locally Convex Space, in: Proc. of Leningrad Engineering-Economic Institute, Vol. 63 (1966) 13–17 (in Russian).
[10] H. Rådström. An Embedding Theorem for Spaces of Convex Sets, Proc. Amer. Math. Soc. 3 (1952), 165–169.
[11] H. Ratschek, G. Schröder. Über den quasilinearen Raum, in: Ber. Math.-Statist. Sekt., Vol. 65 (Forschungszentr. Graz, 1976) 1–23.
[12] H. Ratschek, G. Schröder. Representation of Semigroups as Systems of Compact Convex Sets, *Proc. Amer. Math. Soc.* **65** (1977) 24–28.
[13] R. T. Rockafellar. Convex Analysis, Princeton, 1970.
[14] R. Schneider. *Convex Bodies: The Brunn-Minkowski Theory* (Cambridge Univ. Press, 1993).
[15] W. Weil. Über den Vektorraum der Differenzen von Stützfunktionen konvexer Körper, *Math. Nachr.* **59** (1974) 353–369.
[16] G. Ziegler. Lectures on Polytopes, Springer, 1995.

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ВЪРХУ ЛИНЕЙНИ КОМБИНАЦИИ ОТ СИМЕТРИЧНИ СЕГМЕНТИ

Й. Акълдъз, Д. Клаудио, С. Марков

Изследвани са линейни комбинации от симетрични сегменти в равнината с помощта на системата за компютърна алгебра MATHEMATICA.