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CONFORMAL C-NETS AND CONFORMAL B-NETS IN A THREE-DIMENSIONAL RIEMANNIAN SPACE [*]

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In an n-dimensional Weyl space c-nets and b-nets are defined, making use of Chebyshevian and geodesic curvatures of the lines of an arbitrary net in [4]. Conformal c and b nets in a Weyl space are studied by Zlatanov [5]. The conformal geometry of the compositions determined by the normalized net in a three-dimensional Weyl space is studied in [1].
In this paper c and b nets are defined in a three-dimensional Riemannian space. There are obtained the necessary and sufficient conditions for a net to be conforming-c or conforming-b, and there are found some characteristics of the Riemannian spaces, containing these nets.

1. Preliminaries. The conformal transformation:

$$\overset{*}{g}_{is} = e^{2\lambda} g_{is} \tag{1.1}$$

transforms the net $\left(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}\right) \in V_3(g_{is})$ into $\left(\underset{1}{\overset{*}{v}}, \underset{2}{\overset{*}{v}}, \underset{3}{\overset{*}{v}}\right) \in \overset{*}{V}_3(\overset{*}{g}_{is})$. The fields of directions of the net $\left(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}\right)$ are transformed [3, p.125] so:

$$\underset{\alpha}{\overset{*}{v}}{}^i = e^{-\lambda} \underset{\alpha}{v}{}^i, \quad \overset{\alpha}{\overset{*}{v}}_i = e^{\lambda} \overset{\alpha}{v}_i. \tag{1.2}$$

The vector of the conformal transformation is $\lambda_k = \partial_k \lambda = \dfrac{\partial \lambda}{\partial u^k}$ [2, p. 162].

Let Γ^i_{ks} and $\overset{*}{\Gamma}{}^i{}_{sk}$ be the coefficients of connections in the spaces V_3 and $\overset{*}{V}_3$. Let ∇ and $\overset{*}{\nabla}$ be the covariant derivatives of the connections Γ^i_{ks} and $\overset{*}{\Gamma}{}^i{}_{ks}$ respectively, i.e.

$$\nabla_k \underset{\alpha}{v}{}^i = \partial_k \underset{\alpha}{v}{}^i + \Gamma^i_{ks} \underset{\alpha}{v}{}^s, \ \overset{*}{\nabla}_k \underset{\alpha}{v}{}^i = \partial_k \underset{\alpha}{v}{}^i + \overset{*}{\Gamma}{}^i{}_{ks} \underset{\alpha}{v}{}^s.$$

If the coefficients of the derivative equations in V_3 are denoted by $\underset{\alpha}{\overset{\sigma}{T}}_k$, then we have:

$$\partial_k \underset{\alpha}{v}{}^i + \Gamma^i_{ks} \underset{\alpha}{v}{}^s = \underset{\alpha}{\overset{\sigma}{T}}_k \underset{\sigma}{v}{}^i, \quad \sigma, s = 1, 2, 3. \tag{1.3}$$

From here we obtain:

$$\underset{\alpha}{\overset{\beta}{T}}_k = \left(\partial_k \underset{\alpha}{v}{}^i + \Gamma^i_{ks} \underset{\alpha}{v}{}^s\right) \overset{\beta}{v}_i. \tag{1.4}$$

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It is known that [2, p. 161]:

$$\overset{*}{\Gamma}{}^i_{ks} = \Gamma^i_{ks} + \delta^i_s\lambda_k + \delta^i_k\lambda_s - g^{ij}g_{sk}\lambda_j. \tag{1.5}$$

Let us denote the coefficients of the derivative equations in the space $\overset{*}{V}_3$ by $\underset{\alpha}{\overset{\sigma}{P}}_k$. Then $\overset{*}{\nabla}_k \underset{\alpha}{v}^i = \underset{\alpha}{\overset{\sigma}{P}}_k \underset{\sigma}{v}^i$. From (1.2), (1.4) and (1.5), we obtain:

$$\underset{\alpha}{\overset{\beta}{P}}_k = \underset{\alpha}{\overset{\beta}{T}}_k + \lambda_s \left(\overset{\beta}{v}_k \underset{\alpha}{v}^s - \overset{\beta}{v}_i g^{is} \underset{\alpha}{v}_k \right), \quad \alpha \neq \beta. \tag{1.6}$$

2. Conformal *c*-nets in V_3. The net $\left(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}\right) \in V_3(g_{is})$ will be called a **c**-net if [4]:

$$\sum_{\alpha=1}^{3} \underset{\alpha}{\overset{\sigma}{\imath}} \underset{\sigma}{v}^i = 0, \quad \underset{\alpha}{\overset{\sigma}{\imath}} = \underset{\alpha}{\overset{\sigma}{T}}_k \underset{\alpha}{v}^k, \quad \alpha, \sigma = 1, 2, 3. \tag{2.1}$$

The quantity $\underset{\alpha}{\overset{\sigma}{\imath}}$ is called a *geodesic curvature* of the line $(\underset{\alpha}{v})$.

Proposition. *The net* $\left(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}\right) \in V_3$ *is a* ***c****-net when*

$$\sum_{\alpha=1}^{3} \nabla_{[j} \underset{\alpha}{v}^s \Gamma^k_{s]k} = 0. \tag{2.2}$$

Proof. Let us consider a *c*-net-$\left(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}\right) \in V_3$. From the equation $\sum_{\alpha=1}^{3} \underset{\alpha}{\overset{\sigma}{\imath}} \underset{\sigma}{v}^i = 0$, taking into account (2.1) and (1.4), we obtain $\sum_{\alpha=1}^{3} (\partial_k \underset{\alpha}{v}^i \underset{\alpha}{v}^k + \Gamma^i_{ks} \underset{\alpha}{v}^s \underset{\alpha}{v}^k) = 0$.From here, by a contraction with $\overset{\alpha}{v}_i$, we find:

$$\sum_{\alpha=1}^{3} (\partial_k \underset{\alpha}{v}^k + \Gamma^k_{sk} \underset{\alpha}{v}^s) = 0. \tag{2.3}$$

Applying the integrability condition of (2.3) and taking into account that $\Gamma^k_{sk} =$ grad, we immediately obtain the proposition. □

Definition 2.1. *A net* $\left(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}\right) \in V_3(g_{is})$*, admitting a conformal transformation into a* ***c****-net, is called a conformal* ***c****-net.*

Let the net $\left(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}\right) \in V_3(g_{is})$ be transformed into a **c**-net $\left(\underset{1}{\overset{*}{v}}, \underset{2}{\overset{*}{v}}, \underset{3}{\overset{*}{v}}\right) \in \overset{*}{V}_3(\overset{*}{g}_{is})$ by the conformal transformation (1.1). According to (2.1), the following relations are valid:

$$\sum_{\alpha=1}^{3} \underset{\sigma}{\overset{*}{v}}{}^i \underset{\alpha}{\overset{\sigma *}{\imath}} = \sum_{\alpha=1}^{3} e^{-2\lambda} \left(\underset{\alpha\sigma}{\overset{\sigma}{\imath}} v^i + \lambda_s \left(\underset{\alpha}{v}^i \underset{\alpha}{v}^s - g^{is} \right) \right) = 0, \quad \sigma = 1, 2, 3. \tag{2.4}$$

Consequently, the vector of the conformal transformation λ_s will satisfy:

$$\sum_{\alpha=1}^{3} \lambda_s \left(g^{is} - \underset{\alpha}{v}^i \underset{\alpha}{v}^s \right) = \sum_{\alpha=1}^{3} \underset{\sigma}{v}^i \underset{\alpha}{\overset{\sigma}{\imath}}, \quad \sigma = 1, 2, 3. \tag{2.5}$$

Theorem 2.1. *A non-orthogonal net* $\left(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}\right) \in V_3$ *is a conformal c-net if and only* *if:*

$$a)\sum_{\alpha=1}^{3}\left(\partial_k \underset{\alpha}{v}^i \underset{\alpha}{v}^k + \Gamma^i_{kj}\underset{\alpha}{v}^k \underset{\alpha}{v}^j + \lambda_s\left(\underset{\alpha}{v}^s \underset{\alpha}{v}^i - g^{is}\right)\right) = 0, \quad b)\frac{D_k \overset{k}{v}_s}{D} = \lambda_s = \text{grad},$$

where:

$$D = 54\cos\underset{12}{\omega}\cos\underset{23}{\omega}\cos\underset{31}{\omega}, \qquad D_\alpha = 9\cos\underset{\beta\gamma}{\omega}\left(-a_\alpha\cos\underset{\beta\gamma}{\omega} + a_\beta\cos\underset{\gamma\alpha}{\omega} + a_\gamma\cos\underset{\alpha\beta}{\omega}\right),$$

$$(\alpha,\beta,\gamma) = (1,2,3)\,,\ (2,3,1)\,,\ (3,1,2)\,.$$

and $\underset{\alpha\beta}{\omega}$ *denote the angles between the fields of directions* $\underset{\alpha}{v}^i$ *and* $\underset{\beta}{v}^i$ *of the net.*

Proof. 1. The conformal transformation (1.1) transforms the net $\left(\underset{1}{v},\underset{2}{v},\underset{3}{v}\right) \in V_3$ into the **c**-net $\left(\underset{1}{\overset{*}{v}},\underset{2}{\overset{*}{v}},\underset{3}{\overset{*}{v}}\right) \in \overset{*}{V}_3(\overset{*}{g}_{is})$. From (2.4), taking into account the representation of the geodesic curvature and (1.4), we find a). By contracting (2.5) with $\overset{1}{v}_i, \overset{2}{v}_i$ and $\overset{3}{v}_i$ respectively, we obtain:

$$\lambda_s \sum_{\alpha=1}^{3}\left(g^{is}\overset{m}{v}_i - \delta^m_\alpha \underset{\alpha}{v}^s\right) = \sum_{\alpha=1}^{3}\underset{\alpha}{\overset{m}{\imath}} = a_m,\ m = 1,2,3. \tag{2.6}$$

We introduce the denotations $\sum_{\alpha=1}^{3}\underset{\alpha}{\overset{m}{\imath}} = a_m, m = 1,2,3$. Let the vector of the conformal transformation λ_s be expressed as $\lambda_s = x\overset{1}{v}_s + y\overset{2}{v}_s + z\overset{3}{v}_s$. The coefficients x, y and z satisfay the sistem:

$$3y\cos\underset{12}{\omega} + 3z\cos\underset{13}{\omega} = a_1, 3x\cos\underset{12}{\omega} + 3z\cos\underset{23}{\omega} = a_2, 3x\cos\underset{13}{\omega} + 3y\cos\underset{23}{\omega} = a_3. \tag{2.7}$$

If we use D to denote the determinant of the coefficients of the unknown variables in this system, and $D_\alpha(\alpha = 1,2,3)$ for the determinants obtained after substituting the a column by the column of the free members in D, we obtain:

$$D = 54\cos\underset{12}{\omega}\cos\underset{23}{\omega}\cos\underset{31}{\omega}, \qquad D_\alpha = 9\cos\underset{\beta\gamma}{\omega}(-a_\alpha\cos\underset{\beta\gamma}{\omega} + a_\beta\cos\underset{\gamma\alpha}{\omega} + a_\gamma\cos\underset{\alpha\beta}{\omega}),$$

$$(\alpha,\beta,\gamma) = (1,2,3),\ (2,3,1),\ (3,1,2).$$

Since the net $\left(\underset{1}{v},\underset{2}{v},\underset{3}{v}\right) \in V_3(g_{is})$ is not orthogonal, then $D \neq 0$. Therefore, (2.7) has a unique solution for the vector of the conformal transformation $\lambda_s = \frac{1}{D}(D_1\overset{1}{v}_s + D_2\overset{2}{v}_s + D_3\overset{3}{v}_s)$.

The vector of the conformal transformation λ_s is gradient [2, p. 162], i.e.

$$\frac{1}{D}(D_1\overset{1}{v}_s + D_2\overset{2}{v}_s + D_3\overset{3}{v}_s) = \text{grad}.$$

2. Conversely, let a) and b) hold for the net $\left(\underset{1}{v},\underset{2}{v},\underset{3}{v}\right) \in V_3(g_{is})$. From **a**), taking into account (1.3) and (1.6), we get (2.4). It means that $\left(\underset{1}{\overset{*}{v}},\underset{2}{\overset{*}{v}},\underset{3}{\overset{*}{v}}\right) \in \overset{*}{V}_3(\overset{*}{g}_{is})$ is a *c*-net. From **b**) following the inverse way of argument, we obtain that the coefficients λ_s satisfy system (2.7). It has a unique solution, i.e. the net $\left(\underset{1}{v},\underset{2}{v},\underset{3}{v}\right) \in V_3(g_{is})$ is a conformal *c*-net. □

3. Conformal *b*-nets in V_3. Following [4], we shall call a $\left(\underset{1}{v},\underset{2}{v},\underset{3}{v}\right) \in V_3\,(g_{is})$ net a *b*-net if the following condition holds:

$$\sum_{\alpha=1}^{3} \overset{\alpha}{b}_i = 0, \tag{3.1}$$

where

$$\overset{\alpha}{b}_i = \overset{\alpha\beta}{\underset{\beta}{\rho}} v_i, \quad \overset{\alpha}{\underset{\beta}{\rho}} = \overset{(\alpha)}{T_k} \underset{\beta\,(\alpha)}{v}{}^{k}, \quad \alpha,\beta = 1,2,3. \tag{3.2}$$

The quantity $\overset{\alpha}{\underset{\beta}{\rho}}$ is called a Chebyshevian curvature of second kind. (The bracket indices are not to be summed).

Definition 3.1. *A net* $\left(\underset{1}{v},\underset{2}{v},\underset{3}{v}\right) \in V_3$*, admitting a conformal transformation into a* ***b****-net will be called a conformal* ***b****-net.*

Theorem 3.1. *The net* $\left(\underset{1}{v},\underset{2}{v},\underset{3}{v}\right) \in V_3$ *is a conformal* ***b****- net if and only if:*

$$a)\quad \partial_k \underset{\alpha}{v}{}^{k} \overset{\alpha}{v}_i + \Gamma^k_{ki} + 2\lambda_i = 0, \qquad b)\quad \lambda_s = \widetilde{z}^{\alpha}_{s} b_\alpha = \text{grad},$$

where $z^s_\alpha = -3\underset{\alpha}{v}{}^{s} + g^{ks}\left(\overset{\alpha}{v}_k + \overset{\beta}{v}_k \cos \underset{\alpha\beta}{\omega} + \overset{\gamma}{v}_k \cos \underset{\alpha\gamma}{\omega}\right)$, $z^s_\alpha \widetilde{z}^{\alpha}_{k} = \delta^s_k$ *and* $b_\alpha = \overset{\alpha}{\underset{\alpha}{\rho}} + \overset{\beta}{\underset{\alpha}{\rho}} + \overset{\gamma}{\underset{\alpha}{\rho}}$.

Proof. 1. Let the conformal transformation (1.1) transorm the net $\left(\underset{1}{v},\underset{2}{v},\underset{3}{v}\right) \in V_3$ into a *b*-net $\left(\underset{1}{\overset{*}{v}},\underset{2}{\overset{*}{v}},\underset{3}{\overset{*}{v}}\right) \in \overset{*}{V}_3(\overset{*}{g}_{is})$. From (3.1) follows that $\sum_{\alpha=1}^{3} \overset{\overset{\alpha}{*}}{b}_i = 0$. Taking into account (1.4), (1.5) and (1.6), we obtain **a**). From the conditions (3.2), (1.2) and (1.6) for the vector of the conformal transformation λ_s we have:

$$\sum_{\alpha=1}^{3} \left(\overset{\alpha}{\underset{\sigma}{\rho}} + \lambda_s (\underset{\sigma}{v}{}^{s} - \overset{(\alpha)}{v}{}_k g^{ks} \cos \underset{(\alpha)\sigma}{\omega}) \right) \overset{\sigma}{v}_i = 0. \tag{3.3}$$

After contracting (3.3) by $\overset{1}{v}_i, \overset{2}{v}_i$ and $\overset{3}{v}_i$ respectively, we arrive at

$$\lambda_s[-3\underset{\alpha}{v}{}^{s} + g^{ks}(\overset{\alpha}{v}_k + \overset{\beta}{v}_k \cos \underset{\alpha\beta}{\omega} + \overset{\gamma}{v}_k \cos \underset{\alpha\gamma}{\omega})] = \overset{\alpha}{\underset{\alpha}{\rho}} + \overset{\beta}{\underset{\alpha}{\rho}} + \overset{\gamma}{\underset{\alpha}{\rho}}, \tag{3.4}$$

where $(\alpha,\beta,\gamma) = (1,2,3),\ (2,3,1),\ (3,1,2)$.

We introduce the denotations:

$$z^s_\alpha = -3\underset{\alpha}{v}{}^{s} + g^{ks}(\overset{\alpha}{v}_k + \overset{\beta}{v}_k \cos \underset{\alpha\beta}{\omega} + \overset{\gamma}{v}_k \cos \underset{\alpha\gamma}{\omega}), b_\alpha = \overset{\alpha}{\underset{\alpha}{\rho}} + \overset{\beta}{\underset{\alpha}{\rho}} + \overset{\gamma}{\underset{\alpha}{\rho}}.$$

Then (3.4) takes the form $\lambda_s z^s_\alpha = b_\alpha$. The matrix (z^s_α) is nonsingular and there exists an inverse one of it $(\tilde{z}^{\alpha}_{k})$. From (3.4) for the vector of the conformal transformation we obtain $\lambda_s = \tilde{z}^{\alpha}_{s} b_\alpha$.The vector of the conformal transformation is gradient.

2. Conversely, let **a**) and **b**) hold for the net $\left(\underset{1}{v},\underset{2}{v},\underset{3}{v}\right) \in V_3$. From a), taking into account (1.3), (1.5) and (1.6), we find (3.1) and (3.2), which mean that the net $\left(\underset{1}{v},\underset{2}{v},\underset{3}{v}\right) \in V_3$ is a *b*-net. From **b**), follwing the inverse way of argument, we obtain that the net $\left(\underset{1}{v},\underset{2}{v},\underset{3}{v}\right) \in V_3$ admits a conformal transformation into a *b*-net $\left(\underset{1}{\overset{*}{v}},\underset{2}{\overset{*}{v}},\underset{3}{\overset{*}{v}}\right) \in \overset{*}{V}_3(\overset{*}{g}_{is})$. $\square$

The curvature coordinates of the points in the space V_3 will be denoted with $u^1 = u$, $u^2 = v$, $u^3 = w$. If we choose an arbitrary net $\left(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}\right)$ as a coordinate one of V_3, then the fundamental form of the space will be:

$$ds^2 = A^2 du^2 + B^2 dv^2 + C^2 dw^2 + 2AB \cos \underset{12}{\omega} dudv + 2AC \cos \underset{13}{\omega} dudw + 2BC \cos \underset{23}{\omega} dvdw,$$

where A, B and C are functions of u, v and w.

The unit fields of directions of the net and their mutual ones have the coordinates:

$$\underset{1}{v}\left(\frac{1}{A}, 0, 0\right), \ \underset{2}{v}\left(0, \frac{1}{B}, 0\right), \ \underset{1}{v}\left(0, 0, \frac{1}{C}\right), \ \overset{1}{v}(A, 0, 0), \ \overset{2}{v}(0, B, 0), \ \overset{3}{v}(0, 0, C). \tag{3.5}$$

Theorem 3.2. *The fundamental form of the space $V_3(g_{is})$ in the parameters of a conformal **b**-coordinate net has the form:*

(3.6)

$$ds^2 = A^2 \left[\begin{array}{c} du^2 + f(v,w)\psi(u,w) dv^2 + \xi(u,v)\eta(u,w) dw^2 + 2f(v,w)\psi(u,w) dudv + \\ 2\xi(u,v)\eta(u,w) dudw + 2f(v,w)\xi(u,v)\tau(u,w) dvdw \end{array} \right],$$

where $f(v,w)$, $\psi(u,w)$, $\xi(u,v)$, $\eta(u,w)$, $\tau(u,w)$ are arbitrary functions.

Proof. Let the net $\left(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}\right) \in V_3$ be a conformal b-net. From $\partial_k \underset{\alpha}{v}^k \overset{\alpha}{v}_i + \Gamma_{ki}^k + 2\lambda_i = 0$, and taking into account that $\Gamma_{ki}^k = \text{grad}$, $\lambda_i = \text{grad}$ follows that $\partial_k \underset{\alpha}{v}^k \overset{\alpha}{v}_i = \text{grad}$. If we choose the net $\left(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}\right) \in V_3$ as a coordinate one, then from the last equation we find: $\partial_1 \underset{1}{v}^1 \overset{1}{v}_1 + \partial_2 \underset{2}{v}^2 \overset{2}{v}_2 + \partial_3 \underset{3}{v}^3 \overset{3}{v}_3 = \text{grad}$. From here, taking into account (3.5), we obtain: $\left((\ln A)_u, (\ln B)_v, (\ln C)_w\right) = \text{grad}$. It means that $B = f(v,w)\psi(u,w)A$ and $C = \xi(u,v)\eta(u,w)B$. Then the fundamental form of the space V_3 takes the form (3.6). □

Obviously, when $f(v,w) = \psi(u,w) = \xi(u,v) = \eta(u,w) = 1$ the space V_3 is conformally Euclidean.

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КОНФОРМНО *C* МРЕЖИ И КОНФОРМНО *B* МРЕЖИ В ТРИМЕРНО РИМАНОВО ПРОСТРАНСТВО

Иван Ат. Бадев

Използвайки Чебишева и геодезично кривина на линия на произволна мрежа, c и b мрежите в n-мерно пространство на Вайл са дефинирани и изследвани в [4]. Златанов изучава конформно c и b мрежи в пространство на Вайл в [5]. Конформната геометрия на композиция породени от мрежи в тримерно пространство на Вайл е разгледана в [1].

В тази работа са изучавани c и b мрежи в тримерно риманово пространство. Получени са необходими и достатъчни условия дадена мрежа да е конформно c мрежа или конформно b мрежа. Намерени са характеристики на риманови пространства, съдържащи тези мрежи.