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ANALYTIC BERNOULLI FIXED POINTS FREE FLOW ON THE 3-SPHERE*

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A real analytic Bernoulli fixed points free flow on the 3-sphere is constructed (Corollary 1). This answers affirmatively a question of Harrison and Pugh [9]. In fact, such a flow is found on a larger class of 3-manifolds (Theorem 2). The construction relies on a famous example of Burns and Gerber [5] of Riemannian surfaces with Bernoulli geodesic flow.

Introduction. In their paper [9] Harrison and Pugh asked whether there exists a real analytic ergodic *fixed points free* flow on the 3-sphere $\mathbb{S}^3$. In the present note we answer affirmatively this question. Moreover, the constructed flow is Bernoulli, which is a metrical property, much stronger than ergodicity. In fact, we find fixed points free analytic Bernoulli flows on a larger class of 3-manifolds. This class may be described as follows:

Definition 1. *Let S be a closed Riemannian 2-surface and T^1S denotes the fiber bundle of length 1 tangent vectors to S, which is a 3-manifold. Consider the class $\mathfrak{F}(S)$ of all connected covering spaces with base T^1S with a finite holonomy group. Then we define the class $\mathfrak{F} = \cup\mathfrak{F}(S)$, where S runs over all such 2-surfaces.*

We shall show that any manifold of class $\mathfrak{F}$ carries such a flow. We prove in this note the following two facts:

Let M be a 3-manifold from class $\mathfrak{F}$. Then there exists a real analytic Bernoulli fixed points free flow on M.

There is a real analytic Bernoulli (*with respect to Lebesgue measure*) *fixed points free flow on the 3-sphere S^3.*

Let us make some remarks: Each manifold of class $\mathfrak{F}$ is compact, as the holonomy group is supposed to be finite. The class $\mathfrak{F}$ includes all manifolds of the form T^1S, since they can be regarded as a trivial covering with base T^1S. The following classical 3-manifolds belong to class $\mathfrak{F}$:

the 3-dimensional sphere $\mathbb{S}^3$ – indeed, the space $T^1\mathbb{S}^2$ is known to be homeomorphic to the real projective space $\mathbb{RP}^3$, and there is a natural double covering of $\mathbb{S}^3$ onto $\mathbb{RP}^3$,

the 3-dimensional torus $\mathbb{T}^3$, as it is obviously homeomorphic to $T^1\mathbb{T}^2$.

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The construction relies on a famous example of Burns and Gerber [5] of a Riemannian metric on any closed surface, whose geodesic flow is Bernoulli with respect to its Liouville measure.

Overview of the problem. The problem concerning the existence of flows with different metrical or topological properties on 3-manifolds is a classical one. One of the first results in that area was Oxtoby and Ulam's example [14] of *topologically transitive* and ergodic continuous authomorphism of strongly connected polyhedra. Let us note also Besicovitch's explicit example [2] of transitive authomorphisms of the plane with a single fixed point. Later Sidorov [15] made this example class C^{∞}. Another famous article is [1], where Anosov and Katok construct C^{∞} ergodic flows on 3-manifolds admitting a smooth effective $\mathbb{S}^1$-action. These flows are not fixed points free. The authors in fact construct here not only ergodic flows, but a large series of examples with different metrical properties (such as mixing properties, prescribed spectrum, etc.). Later Katok [10] modified their method to find a *minimal* diffeomorphism in $\mathbb{S}^3$ (in fact in any principal $\mathbb{S}^1$-bundle). Blokhin [3] constructed smooth ergodic flows on any closed 2-surface, except for the 2-sphere, the projective plane and Klein's bottle, where they are impossible.

In [11] Katok constructed a C^{∞} Bernoulli authomorphism of any compact 2-surface. Gerber [7] has shown that analytic examples like Katok's do exist. Starting from Katok's example, Harrison and Pugh constructed in [9] a smooth ergodic (with respect to Lebesgue measure) fixed points free flow on $\mathbb{S}^3$ (as well as on any lens space). Here they asked some questions. One of them is whether there exists a real analytic ergodic fixed points free flow on $\mathbb{S}^3$. We answer affirmatively this question in the present note. Let us notice that analyticity is not just a technical detail, as it often imposes nontrivial topological obstructions on flows and foliations (see for example [16]).

There are some other interesting topics which are situated very closely to the discussed one. They are focused mainly on Seifert's and Arnold's conjectures where the existence of periodic or stationary orbits is stated. For an overview of these problems see Ginzburg's paper [8]. One of the famous open problems here is Gottshalk's conjecture about the existence of a minimal flow in $\mathbb{S}^3$.

Some basic definitions. By *surface* we mean here a connected closed 2-manifold. A manifold is *closed*, if it is compact and has no boundary.

A measure-preserving authomorphism φ of a space with measure (X, μ) is called *ergodic*, if any φ-invariant measurable set in X has measure 0 or 1. There is a subclass of authomorphisms, called *Bernoulli,* with best (in some sense) mixing properties. For the definition and the properties of Bernoulli authomorphisms see [6]. Each Bernoulli authomorphism is ergodic and has the mixing property. A *flow* is called ergodic (resp. Bernoulli), if the time 1 map of the flow is ergodic (resp. Bernoulli).

A manifold is *real analytic* (or *analytic* for simplicity), if all the coordinate functions are real analytic. The metric g of a Riemannian manifold (M, g) is *analytic*, if there is an analytic atlas on M, such that all the components g_{ij} of the metric tensor are real analytic in the corresponding coordinate chart.

A flow on an n-manifold M is *analytic*, if there is an analytic atlas on M of "flow-boxes", i.e. such that the coordinate functions send the flow into the flow generated by the constant vector field $(1, 0, ..., 0)$ in $\mathbb{R}^n$.

If M is a smooth Riemannian manifold we denote by TM the tangent vector bundle on M and by T^1M the fiber bundle of length 1 tangent vectors. If M is n-dimensional, then TM is $2n$-dimensional, and T^1M is $(2n-1)$-dimensional. Now, if v is an element of TM, the correspondence $v \to (v, 0) \in TTM$ defines a vector field on TM. This field defines a flow in TM. It is easy to see that the manifold T^1M is invariant with respect to this flow. The restriction of the flow to T^1M is called the *geodesic flow* in M. Clearly, this flow is fixed points free.

If M is a 3-manifold, the 1-form ω is a *contact form*, if the 3-form $\omega \wedge d\omega$ is everywhere nondegenerate. A flow in M is called *contact*, if it preserves some contact form.

Statement and proof of the results. As it has been pointed out above, we shall make use of some deep results of Burns and Gerber [4] as well of their example of Bernoulli geodesic flows in [5].

Theorem 1. (Burns, Gerber [4]) *Let M be a class C^3 compact 3-manifold and φ^t be a C^2 flow leaving invariant the measure defined by some C^2 Riemannian metric g on M. Suppose that there is a φ^t-invariant distribution P transverse to the flow. Let K be a continuous family of 2-dimensional cones such that $K(x) \subseteq P(x)$ for all $x \in M$. Suppose that there is a connected open subset U of M such that*

(i) $\cup_{t\in\mathbb{R}}\varphi^t U$ has full measure in M;

(ii) $d\varphi^t(K(x))$ is strictly contained in $K(\varphi^t x)$ whenever $x \in U$, $t > 0$ and $\varphi^t x \in U$.

Then the flow φ^t is ergodic. If in addition φ^t is a contact flow and P is the kernel of the contact form, then φ^t is Bernoulli with respect to its Liouville measure.

The key condition here is (ii) – roughly speaking, the flow φ^t is compressing all the time the cones family $K(x)$ into itself. The authors apply this theorem to their example of compact C^∞ Riemannian surfaces (S, g) with the following property:

The set on which the curvature is nonnegative is an union of disjoint disks $C_1, \dots, C_n$ called “caps”. The boundary of each cap is a geodesic circle, each cap is radially symmetric and its curvature is a nondecreasing function of distance from the boundary. This function has positive derivative at the boundary and vanishes there.

It turns out that the geodesic flow on S satisfies the assumptions in Theorem 1 for some natural definition of $P(x)$, $K(x)$ and U. Therefore, the geodesic flow on S is Bernoulli.

Furthermore, Burns and Gerber prove in [5] that if $\widetilde{g}$ is a C^3 Riemanniam metric close enough to g in the C^2 topology, then the corresponding geodesic flow $\varphi^t_{\widetilde{g}}$ on $T^{1,\widetilde{g}}S$ is ergodic and Bernoulli with respect to its Liouville measure. Finally they show that there exists on S a real analytic metric $\widetilde{g}$ close enough to g. Therefore $T^{1,\widetilde{g}}S$ becomes a real analytic 3-manifold with a fixed points free Bernoulli flow, namely – the geodesic flow.

As we explained in the introduction, we shall extend this result to connected coverings with base $T^{1,\widetilde{g}}S$ and a finite holonomy group. This includes the important case of the 3-sphere $\mathbb{S}^3$.

Let M be such a covering and $p : M \to T^{1,\widetilde{g}}S$ be the projection. Let us note that M is compact, since every fiber is finite. Now, roughly speaking, we shall follow the construction in [5] and taking pull back of forms, measures and sets in M, we verify that the conditions of Theorem 1 are fulfilled in M for the pull back of the geodesic flow in $T^{1,\widetilde{g}}S$.

The set U is defined as follows: For each cap C_i a closed disk D_i is taken so that $D_1, \ldots, D_n$ are pairwise disjoint, $C_i \subset Int\, D_i$ and (D_i, g) is radially symmetric about the centre of (C_i, g). Then U is defined as the set of vectors in $T^{1,\widetilde{g}}S$, whose footpoints lie outside $\cup D_i$. Furthermore, it is shown that the set U satisfies condition (i) from Theorem 1 for $M = T^{1,\widetilde{g}}S$ and φ^t being the geodesic flow. Moreover, a φ^t-invariant distribution P transverse to the flow, as well as a cones family $K(x) \subseteq P(x)$ is constructed and it is shown that condition (ii) from Theorem 1 is fulfilled. Finally, it is shown that φ^t is preserving some contact 1-form ω.

We shall show that the same is the situation in M. First of all, we shall consider in M the pull-back of the measure in $T^{1,\widetilde{g}}S$ with the natural normalization: $\mu^*(A) = 1/m\mu(p(A))$, where A is not containing two points from one and the same fiber and m is the cardinality of the fiber. The projection $p : M \to T^{1,\widetilde{g}}S$ induces a natural map between the tangent bundles $dp : TM \to TT^{1,\widetilde{g}}S$. Note that since p is a covering, the differential dp is an isomorphism on each fiber. Then we may define correctly $P'(x) = (dp)^{-1}(P(p(x)))$, $K'(x) = (dp)^{-1}(K(p(x)))$, $U' = p^{-1}(U)$ and ψ^t as the pull-back under p of the flow φ^t. Then condition (ii) is trivially fulfilled, since for any $x \in p^{-1}(U)$ we have $d\psi^t K'(x) = d\psi^t (dp)^{-1}(K(p(x))) = d(dp)^{-1}\varphi^t K(p(x)) = (dp)^{-1} d\varphi^t K(p(x)) \subset (dp)^{-1} K(\varphi^t p(x))$. On the other hand, $K'\psi^t(x) = (dp)^{-1}(K(p(\psi^t(x)) = (dp)^{-1}(K\varphi^t p(x))$. Thus, for each $x \in U' = p^{-1}(U)$ we have $d\psi^t K'(x) \subset K'\psi^t(x)$, so condition (ii) is fulfilled in M.

Condition (i) is also trivially fulfilled: $\cup_{t\in\mathbb{R}}\psi^t U' = \cup_{t\in\mathbb{R}}\psi^t p^{-1}(U) = \cup_{t\in\mathbb{R}} p^{-1}(\varphi^t(U)) = p^{-1}(\cup_{t\in\mathbb{R}}\varphi^t(U))$, but the set $\cup_{t\in\mathbb{R}}\varphi^t(U)$ has full measure in $T^{1,\widetilde{g}}S$, hence $\cup_{t\in\mathbb{R}}\psi^t U'$ has full measure in M as well.

Now, we have to show that the set $U' = p^{-1}(U)$ is connected, which is almost obvious. Let $r : T^{1,\widetilde{g}}S \to S$ be the natural projection $r(x, v) = x$. Then $rp : M \to S$ is a fiber bundle with each fiber being a disjoint union of m circles. Clearly $U' = M \setminus \cup\, p^{-1}(D_i)$, where D_i are the disjoint closed disks in S from the definition of U. Now, shrinking these disks by a homotopy into different points $q_i \in S$, it follows that U' is homotopically equivalent to $M \setminus \cup\, p^{-1}(q_i)$. But M is a 3-manifold, which is connected by assumption and $\cup p^{-1}(q_i)$ is an union of topological circles, hence $M \setminus \cup\, p^{-1}(q_i)$ is a connected set. Therefore $U' = M \setminus \cup\, p^{-1}(D_i)$ is connected as well.

It is easy to check now that the flow ψ^t is contact and P' is the kernel of the contact form $p^*(\omega)$, where ω is the contact form on $T^{1,\widetilde{g}}S$ with kernel P. Indeed, $p^*(\omega) \wedge dp^*(\omega) = p^*(\omega \wedge d\omega)$ is a nondegenerate everywhere 3-form, so $p^*(\omega)$ is a contact form preserved by the flow ψ^t, as it is the pull-back of φ^t which is preserving ω. The fact that P' is the kernel of $p^*(\omega)$ follows immediately from its definition.

In this way, we checked that all assumptions of Theorem 1 are fulfilled for (M, ψ^t). Therefore, the flow ψ^t is Bernoulli. Recall now that the metric $\widetilde{g}$ is real analytic.

So, we have proved the following

Theorem 2. *Let M be a 3-manifold from class $\mathfrak{F}$ (see Definition 1). Then there exists a real analytic Bernoulli fixed points free flow on M.*

Remark. The flow ψ^t in Theorem 1 in fact is preserving some measure on M with a smooth everywhere positive density.

Indeed, from the construction in [5] it follows that the 3-form $\omega \wedge d\omega$ is everywhere nondegenerate and defines the same measure on TS as the Riemannian metric on S does.

The measure in $T^{1,\tilde{g}}S$ is the restriction of the measure in TS, so it also has a smooth everywhere positive density. But there is a theorem of Moser [13], claiming that each such measure is equivalent to a constant multiple of the Lebesgue measure on M. In this way we may assume that the flow is respecting Lebesgue measure in M.

Corollary 1. *There is a real analytic Bernoulli* (*with respect to Lebesgue measure*) *fixed points free flow on the 3-sphere* $\mathbb{S}^3$.

Proof. Indeed, the space $T^1\mathbb{S}^2$ is known to be homeomorphic to the real projective space $\mathbb{RP}^3$ (c.f. [12]), and there is a double covering of $\mathbb{S}^3$ with base $\mathbb{RP}^3$, hence $\mathbb{S}^3$ belongs to class $\mathfrak{F}$. It remains to apply Theorem 2.

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АНАЛИТИЧЕН БЕРНУЛИЕВ ПОТОК ВЪРХУ 3-МЕРНАТА СФЕРА БЕЗ НЕПОДВИЖНИ ТОЧКИ

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В настоящата работа е построен реално аналитичен Бернулиев поток върху 3-мерната сфера без неподвижни точки (Следствие 1). Това дава положителен отговор на въпрос, поставен от Пю и Харисън в [9]. Всъщност такъв поток е построен върху по-широк клас от тримерни многообразия (Теорема 2). Конструкцията се основава но известния пример на Бърнс и Гербер [5] на риманови повърхнини с Бернулиев геодезичен поток.