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It is a classical result from dimension theory that each n-dimensional compact space contains an n-dimensional Cantor manifold. An important example, after Urysohn, is a minimal compact subset containing an essential n-system. In this note it is shown that each PL pseudomanifold with boundary is minimal (in the sense of Zorn) with respect to some essential system whose frame coincides with its boundary.

Introduction. The attempts to generalize the concept of $n-$dimensionl surface led to the appearance of Cantor manifolds. *Cantor manifold* ($n-$dimensional Cantor manifold; Cantor $n-$manifold or just $\mathcal{C}$-manifolds) is, by definition, every $n-$dimensional compact topological space X, such that for every representation of X as a sum of two proper closed subsets X_1 and X_2 the relation $\dim(X_1 \cap X_2) \geq n-1$ holds.

Urysohn [1] has proved that:

(I) every n-dimensional topological manifold (i.e. locally homeomorphic to $\mathbf{R}^n$) is a Cantor n-manifold;

(II) every absolute boundary (common boundary of several domains in $\mathbf{R}^{n+1}$) is a Cantor n-manifold and

(III) every $n-$dimensional compact Hausdorff space X contains a Cantor n-manifold $Y \subset X$.

Clearly, the Cantor maniflods are intuitively close to the $n-$dimensional surfaces. Note, however, that in the same time the class of Cantor manifolds contains spaces whose structure is far from the notion of the classical topological manifolds. For example, there exists a Cantor $n-$manifold X and a point $x_0 \in X$ which cuts X [2].

That was probably one of the reasons to continue the efforts to define a class of spaces with the propertiess (I) – (III) and which are connected in a stronglier way than Cantor manifolds. Such classes were constructed consecutively by S. Mazurkiewich (the class $\mathcal{M}$) [3], P. Alexandroff (the class $\mathcal{A}$) [2], N. Hadjiivanov (the class $\mathcal{H}$) [4]. Thus we have ordered by inclusion (c.f. [3]) classes of manifolds $\mathcal{A} \subset \mathcal{M} \subset \mathcal{H} \subset \mathcal{C}$ [3].

As a matter of fact, we should note, that the same space Y (property (III) of Urysohn) appears as a Cantor $n-$manifold in all mentioned sences. Namely, it turns out the space Y to be a Cantor n-manifold, an $n-$dimensional manifold in the sence of Alexandroff (so-called (V^n)-continuum), etc.

This observation suggests an idea to define a strongest $n-$dimensional manifold as a membrane which is spanned on some $n-$frame (see definitions below). Then the following question arises: do the properties (I) and (II) hold true for membranes?

This note contains an occasional answer to the first question for pseudomanifolds. More precisely, in this note, we shall prove that each connected (triangulated) manifold with boundary is a *membrane,* i.e. it carries a minimal essential system. Moreover, this is true for a class of polyhedra, which is slightly more general than manifolds, namely – strongly connected *pseudomanifolds* with boundary. We prove that any such polyhedron is a membrane. Let us note that the class of pseudomanifolds includes for example all cones over manifolds with boundary, which are far from being manifolds themselves. Furthermore, if we perform an appropriate factorization in the boundary, the factor space turns out to be a membrane as well. This includes a vast class of spaces, not necessarily manifolds.

This topic has originated from the classical example of a membrane, namely the n-dimensional cube. Alexandroff's Theorem (c.f. [5]) states that the existence of an essential system of range n in X yields $\dim X \geq n$. Then by Zorn's Lemma, one finds in X a minimal essential system of range n.

Definitions and statement of results. The system $\mathcal{S} = \{F_{\pm i}\}$, $i = 1, \dots, n$ of n pairs of closed subsets in X is called *essential system*, if $F_{+i} \cap F_{-i} = \emptyset$ and each system of partitions $\{C_i\}$ between F_{+i} and F_{-i} has a nonempty intersection: $\cap C_i \neq \emptyset$. Further, we refer to the set $F = \cup(F_{-i} \cup F_{+i})$ as *the frame* of the system $\{F_{\pm i}\}$.

A key concept in this paper is the notion for *n-membrane.* The space X is said to be an n-membrane, if there is an essential system $\mathcal{S}$ in X, which is minimal in the following sense: if Y is a proper closed subset of X, then the system $\mathcal{S}_Y = \{Y \cap F_{\pm i}\}$ is no more essential in Y.

A polyhedron P is called *n-pseudomanifold*, if it is a finite union of n-simplexes and each $(n-1)$-simplex is adjacent either to two, or to one n-simplexes. The *boundary* ∂P is defined as the union of all $(n-1)$-simplexes, which are adjacent to one n-simplexs. A pseudomanifold is *strongly connected*, if its $(n-2)$-dimensional skeleton is not dividing P.

Theorem 1. *Let P be a strongly connected n-pseudomanifold with nonempty boundary ∂P. Then P is an n-membrane.*

Corollary 1. *Each connected triangulated n-manifold with nonempty boundary is an n-membrane.*

Proof of the results.

Proof (of Theorem 1) We have to show that the space P carries a minimal essential system. Let $D \subset \partial P$ be some $(n-1)$-dimensional topological disk.

Further, we let I^n to be the n-cube I^n, where I is the interwal $[-1, 1]$. Let also $I^{n-1}_{\pm i}$, $i = 1, \dots, n$, where $I_{\pm i} = \{x | \ x_i = \pm 1\}$ be the system of all $(n-1)$-dimensional faces of the n-cube I^n. According to [6], the system $\{(I^n_{-1}, I^n_{+1}\}$ is essential. Note that the set $\overline{\partial I^n \setminus I^{n-1}_{+1}} = I^{n-1}$ is homeomorphic to D. Let $\varphi : \overline{\partial I^n \setminus I^{n-1}_{+1}} \to D$ be a homeomorphism. Then we define the system $F_{\pm i}$, $i = 1, \dots, n$ in P as follows:

$$F_{\pm i} = \varphi(I^{n-1}_{\pm i}) \quad \text{for all indexes different from } +1 \quad \text{and} \quad F_{+1} = \overline{\partial P \setminus D}.$$

Next we are going to prove that this is a minimal essential system in P. We start by proving that $F_{\pm i}$ is an essential system.

Suppose the contrary. Then in P there exists a system of partitions C_i between F_{+i} and F_{-i} with an empty intersection: $\cap C_i = \emptyset$. Clearly (since P is a normal space) one can find a system of open sets $U_i \supset C_i$ with empty intersection: $\cap U_i = \emptyset$. Moreover, we may suppose that for every k $\overline{U}_k \cap (F_{-k} \cup F_{+k}) = \emptyset$.

Let $P \setminus U_i = A_{+i} \cup A_{-i}$, where $A_{\pm i}$ are closed sets such that $A_{+i} \cap A_{-i} = \emptyset$ and for the interior $\hat{A}_{\pm i}$ of the set $A_{\pm i}$ we have $\hat{A}_{\pm i} \supset F_{\pm i}$. Clearly, $\cup A_{\pm i} = P$. Let us note, that P is metrizable. We shall suppose below, that ϱ is some compatible metric in P. Keeping in mind that the sets $A_{\pm i}$ are compact and $A_{-i} \cap A_{+i} = \emptyset$, we see, that $\varepsilon = \min_i\{\varrho(A_{-i}, A_{+i})\} > 0$

Next, let us take a sufficiently small triangulation τ of $\overline{\partial I^n \setminus I^{n-1}_{+1}}$. Then $\varphi(\tau)$ is a triangulation of D. One can extend $\varphi(\tau)$ to some small triangulation λ of P. We require that for the diameter diam(σ) of every simplex σ of λ the inequality diam$(\sigma) < \varepsilon$ holds. Hence no simplex of λ intersects the sets A_{+i} and A_{-i} simultaneously.

After that, let M be the vertex set of I^n. The points of $N = \varphi(M)$ will be called "base vertices" of the triangulation λ. Clearly, $N \subset D$. Now we shall define $n+1$ closed sets as follows:

$$\Phi_i = A_{+i},\ i = 1, \dots, n \quad \text{and} \quad \Phi_{n+1} = \cup_{i=1}^{n} A_{-i}.$$

Clearly, $\{\Phi_i\}$ is a covering of P. Now we define a "colouring" of the vertices of the triangulation λ with $n+1$ "colours" setting:

$$\nu(a) = \min\{k | a \in \Phi_k\}.$$

It is easy to see that this colouring has the following property: if a is a vertex of λ belonging to the face $F_{\pm i}$, then there is a base vertex b of $F_{\pm i}$, such that $\nu(a) = \nu(b)$. Moreover, this colouring induces a standard "canonical" colouring of the base vertices which may be described as follows. To do this we shall "colour" first the vertices of the n-cube with $n+1$ colours. Let $s = (k_1, \dots, k_n)$ be a vertex of I^n, where $k_i = 0, 1$. Then we set

$$\nu(s) = \max\{i |\, k_1 = \dots = k_i = 1,\ k_{i+1} = 0\} + 1.$$

Clearly, $\nu(s)$ takes $n+1$ values: $1, \dots, n+1$. Then obviously for a base vertex $\varphi(s)$ of λ we have $\nu(\varphi(s)) = \nu(s)$. We shall refer to this special colouring of the base vertices as "canonical colouring".

Now, to obtain the desired contradiction, it suffices to show that there is a full-coloured simplex of λ, i.e. a simplex, whose vertices are coloured with different "colours". This may be done by the argument of Sperner's Lemma. In fact we shall prove that the number of full-coloured simplexes is odd by induction on n. Let us note that the colouring of the base vertices of the face F_{-1} is canonical as well, with colours $2, \dots, n+1$ and each vertex in F_{-1} is coloured with one of these colours. So, let us suppose that the number of full-coloured $(n-1)$-simplexes in F_{-1} is odd. For any n-simplex σ let $k(\sigma)$ denote the number of its $(n-1)$-faces full-coloured with the colours $1, \dots, n$. Consider the number $S = \sum k(\sigma)$, where the sum runs over the n-simplexes of the triangulation. It is easy to see that S is congruent modulo 2 with the number of full-coloured n-simplexes. On the other hand, each interior full-coloured $(n-1)$-simplex is counted in S exactly twice, though each boundary full-coloured $(n-1)$-simplex is counted in S once. So, the number S is congruent modulo 2 with the number of all full-coloured $(n-1)$-simplexes in F_{-1}. But the last number is odd by the induction hypothesis, therefore S is odd as well. So, we proved that there is a full-coloured simplex of λ and now the particular colouring yields that two different vertices of this simplex belong to opposite sets A_{+i} and A_{-i}, which is a contradiction. Therefore the system $F_{\pm i}$ is essential in P.

Now we shall prove the minimality of the system $F_{\pm i}$. Let P' be a proper closed subset of P. Then $P \setminus P'$ contains an open n-ball U lying in $P \setminus \partial P$. Clearly, we may assume the disk D to be situated in some coordinate chart W of P. Then one finds a closed $(n-1)$-disk C lying in W and such that $C \cap \partial P$ is a partition in ∂P between F_{+1} and F_{-1}. But, clearly, C may be moved by means of an isotopy so that its final position intersects U. Hence, we may suppose that C intersects U. This means that $C' = C \cap P'$ is a proper subset of C. Then, since the system $F_{\pm i} \cap C, \quad i = 2, \dots, n$ is a minimal essential system in C, there exist in C' partitions $Z_i, \quad i = 2, \dots, n$ between $F_{+i} \cap C$ and $F_{-i} \cap C$ with an empty intersection: $\cap Z_i = \emptyset$. Now, one extends Z_i to partitions L_i in P' between $F_{+i} \cap P'$ and $F_{-i} \cap P'$ (c.f. [1]). Therefore the sets

$$C' = C \cap P',\ L_i \cap P', \quad i = 2, \dots, n$$

are partitions in P' between $F_{\pm i} \cap P'$ with empty intersection. The theorem is proved.

Now we shall extend the class of membranes, performing some factorization in the boundary of P.

We shall consider factorizations in the boundary ∂P of the following type: Let $\approx$ be a closed equivalence relation such that:

i) The equivalence class of each point $x \in P \setminus \partial P$ is the point x itself.

ii) There is an $(n-1)$-dimensional disk $D \subset \partial P$ such that the class of each point $x \in D$ is the point x itself.

iii) The image of $\partial P \setminus D$ under the equivalence relation has an empty interior in the factor space $P/\approx$.

Theorem 2. *Let P be a strongly connected n-pseudomanifold with a nonempty boundary ∂P and $\approx$ be an equivalence relation of the above type. Then the factor space $K = P/\approx$ is an n-membrane.*

The proof will be omitted, as it is essentially the same as the proof of the previous theorem.

So, under the above considerations, the boundary of a pseudomanifold P is a frame of an essential system $\mathcal{S}$, and P is a membrane of $\mathcal{S}$. This observation suggests us to try to define a *bundary of a polyhedron* as a frame of an essential system.

As an argument in this direction, let us consider the following example:

Example 1. *Consider the right circular cone surface Q with a height 1 and an unit circle S^1 as a base:*

$$Q = \{(x, y, z) | x^2 + y^2 = (1-z)^2;\ 0 \le z \le 1\}$$

and let L be the generator wich connects the points $V(0,0,1)$ and $X(0,0,1)$. Then, consider the map $j : L \to S^1$, defined by

$$j(t, 0, 1-t) = (\cos 2\pi t, \sin 2\pi(1-t), 0).$$

Next we let $K = Q/j$ be the factorspace, obtained by the factorization $p \approx q$ if and only if $j(p) = q$ and $x \approx x$ otherwise. Usually K is known as the so-called "the hat of the jester".

It is easy to see that K is an AR (Absolute Retract) space, which is homeomorphic to a two-dimensional polyhedron. In the same time K has not a boundary in the sence of any homology theory.

To end this paper, let us note that following the abowe considerations, it is easy to verify that K is a membrane with $j(S^1)$ as a frame. Clearly, one may construct similar high-dimensional examples.

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ЕВКЛИДОВИ МЕМБРАНИ

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Класически резултат в теорията на размерностите е, че всеки n-мерен компакт съдържа канторово n-многообразие. Пример за такова от времето на Урисон насам представлява минимален подкомпакт, който съдържа съществена n-система. В тази бележка показваме, че всяко по части линейно псевдомноггобразие с граница е минимален (в смисъл на Цорн) компакт относно някоя съществена система, чиято рамка съвпада с границата му.