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PRICE DYNAMICS IN A TWO REGION MODEL WITH STRATEGIC INTERACTION*

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The paper develops a strategic model of trade between two regions in which, depending on the relation among output, financial resources and transportation costs, the adjustment of prices towards an equilibrium is studied. We derive conditions on the relations among output and financial resources which produce different types of Nash equilibria, as well as the price paths obtained in the process of converging toward an equilibrium.

1. Introduction. The present work develops a model of trade between two regions in which, depending on the relation among output, financial resources and transportation costs, the adjustment of prices towards an equilibrium is studied. We assume that local producers can change prices to balance supply and demand. More specifically, whenever there are unsold quantities left, the price is decreased proportionally and when there are local financial resources unspent, the price is increased proportionally. On the other hand, representative consumers in the two regions seek to maximize their per-period utility in a strategic situation arising from the need to compete for scarce resources.

Under the above setup we derive conditions on the relations among quantities produced and financial resources which produce different types of Nash equilibria, as well as the price paths obtained in the process of converging toward an equilibrium. In certain cases the laws governing price dynamics in discrete time lead to a zero price in one of the regions, which can be interpreted as a breakdown of economic activity in the region.

2. Basic setup and notation. We consider the consumption decisions of two economic agents occupying distinct spatial locations, called region I and II, respectively. The consumer in region I (or, shortly, consumer I) exogenously receives money income $Y_1 > 0$ in each period. Similarly, the consumer in region II (consumer II) receives money income $Y_2 > 0$. For each period t, in region i a fixed quantity $q_i > 0$ of a certain good is supplied at a price $p_{i,t}$. The consumers place orders for the desired quantities in each region, observing their budget constraints and incurring symmetric transportation costs $\rho > 0$ per unit of shipment from the "foreign" region. Each consumer attempts to maximize their total consumption for the current period. Consumers can be considered myopic in that they do not optimize their consumption over a specified time horizon but their decisions are confined only to the current period.

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In cases when total orders for the respective region exceed the quantity available, the following distribution rule is applied: first, the order of the local consumer is executed to the extent possible and then the remaining quantity, if any, is allocated to the consumer from the other region. It is clear then that the choice of orders to be placed has a strategic element to it, since the actual quantity received by the consumer depends on the choices made by the counterpart in the other region. The agents are assumed to have complete knowledge of all the relevant aspects of the situation under discussion.

More formally, for each period t we model the above situation as a static noncooperative game of complete information. Denote by α and β the orders placed by consumer I in region I and II, respectively. In an analogous manner, γ and δ stand for the orders of consumer II in regions I and II, all orders obviously being nonnegative quantities. In period t consumer I's strategy space S_1 is determined by the budget constraint and the nonnegativity restrictions on the orders: $S_1 = \{\alpha p_{1,t} + \beta(p_{2,t} + \rho) \le Y_1, \quad \alpha,\ \beta \ge 0\}$. Consumer II's strategy space in period t is $S_2 = \{\gamma(p_{1,t} + \rho) + \delta p_{2,t} \le Y_2, \quad \gamma,\ \delta \ge 0\}$. Below we adopt the shorthand $p'_{1,t} := p_{1,t} + \rho$ and $p'_{2,t} := p_{2,t} + \rho$. We also omit the subscript t whenever it is evident from the context or irrelevant.

The payoff function for consumer I is given by

$$P_1(\alpha,\ \beta,\ \gamma,\ \delta) = \min(\alpha,\ q_1) + \min(\beta,\ q_2 - \min(\delta,\ q_2)) \tag{1}$$

and that for consumer II by

$$P_2(\alpha,\ \beta,\ \gamma,\ \delta) = \min(\gamma,\ q_1 - \min(\alpha,\ q_1)) + \min(\delta,\ q_2). \tag{2}$$

Any unspent fraction of the current-period income is assumed to perish and consequently the accumulation of stocks of savings is not allowed in the model. At the end of each period, prices are adjusted downwards if the quantity available in the respective region has not been entirely consumed. By the same token, a price is adjusted upwards if a part of the orders placed in the region has not been satisfied. In particular, prices evolve according to the equation

$$\frac{q_i - q_i^{cons}}{q_i} = \frac{p_{i,t} - p_{i,t+1}}{p_{i,t}} \text{ or } p_{i,t+1} q_i = p_{i,t} q_i^{cons} \tag{3}$$

in the former case and follow

$$\frac{Y_i^{res} - p_{i,t} q_i}{p_{i,t} q_i} = \frac{p_{i,t+1} - p_{i,t}}{p_{i,t}} \tag{4}$$

in the latter case. Here q_i^{cons} denotes the total amount consumed in region i and Y_i^{res} stands for the part of the region i's income left unspent. As usual, we consider prices in equilibrium if the price changes in equations (3) and (4) are zero.

For the above model we are interested in two main questions. First, it would be desirable to establish the existence of an equilibrium for the one-period game and specify it in closed form. Second, one would like to be able to trace out the price dynamics entailed by a sequence of one-period games for a given set of initial conditions $p_{10},\ p_{20},\ q_1,\ q_2,\ Y_1,\ Y_2$ and ρ, and characterize their properties.

3. Existence and form of equilibrium. In this section we study the existence and properties of the most popular equilibrium concept – that of Nash equilibrium – for the model specified above, fixing the time period t. Our basic tool for establishing existence is a theorem [1, p. 72] asserting that at least one Nash equilibrium exists for a game of complete information for which (a) the strategy spaces of all players are compact and convex subsets of $\mathbb{R}^m$; (b) all payoff functions are defined, continuous and bounded over the strategy space of the game and (c) any payoff function is quasiconcave in the player's own feasible strategies for a fixed strategy profile of the opponents.

Properties (a) and (b) are immediately verified for our model. To establish property (c) note that the payoff function for each consumer is separable in the consumer's orders and each component of the sum in the payoff is a concave function in the respective order. These observations entail the concavity and hence the quasiconcavity of the payoffs.

Since all the hypotheses of the existence theorem are satisfied for our model, it has at least one Nash equilibrium. We proceed to compute the equilibrium profiles for all possible configurations of $Y_1,\ Y_2,\ q_1,\ q_2,\ p_1,\ p_2$ and ρ. To this end, we derive the best-reply correspondences (see [1, pp. 69-75] for a definition and discussion) for the two consumers, which is straightforward and therefore only the end-results are presented. Table 1 presents the best-reply correspondence for consumer I and Table 2 shows the best-reply correspondence for consumer II. We note in advance that in the course of the price adjustment process one of the prices can become zero, in which case the best reply correspondences take a slightly different form but the same principles apply.

<table>
<tr><td></td><td colspan="2">$\mathbf{A} : \frac{\mathbf{Y_1}}{\mathbf{p_1}} > \mathbf{q_1}$</td><td>$\mathbf{B} : \frac{\mathbf{Y_1}}{\mathbf{p_1}} \leq \mathbf{q_1}$</td></tr>
<tr><td>$\mathbf{I}.\ \mathbf{q_2} - \delta \leq \mathbf{0}$</td><td colspan="2">$q_1 \leq \alpha \leq \frac{Y_1}{p_1},\ 0 \leq \beta \leq \frac{Y_1 - p_1\alpha}{p_2'}$</td><td>$\alpha = \frac{Y_1}{p_1},\ \beta = 0$</td></tr>
<tr><td rowspan="2">$\mathbf{II}.\ \mathbf{q_2} - \delta \in \left(\mathbf{0}, \frac{\mathbf{Y_1}}{\mathbf{p_2'}}\right)$</td><td>$\mathbf{A_1} : \mathbf{0} < \mathbf{q_1} < \frac{\mathbf{Y_1} - \mathbf{p_2'}(\mathbf{q_2} - \delta)}{\mathbf{p_1}}$</td><td>$\mathbf{A_2} : \frac{\mathbf{Y_1} - \mathbf{p_2'}(\mathbf{q_2} - \delta)}{\mathbf{p_1}} \leq \mathbf{q_1} < \frac{\mathbf{Y_1}}{\mathbf{p_1}}$</td><td></td></tr>
<tr><td>$q_1 \leq \alpha \leq \frac{Y_1 - p_2'(q_2 - \delta)}{p_1}$
$q_2 - \delta \leq \beta \leq \frac{Y_1 - p_1\alpha}{p_2'}$</td><td>$(1) : \alpha = q_1,\ \beta = \frac{Y_1 - p_1 q_1}{p_2'}$
$(2) : \frac{Y_1 - p_2'(q_2 - \delta)}{p_1} \leq \alpha \leq q_1,$
$\beta = \frac{Y_1 - p_1\alpha}{p_2'}$
$(3) : \alpha = \frac{Y_1 - p_2'(q_2 - \delta)}{p_1},$
$\beta = q_2 - \delta$</td><td>$(1) : \alpha = \frac{Y_1}{p_1},\ \beta = 0$
$(2) : \alpha = \frac{Y_1 - p_2'\beta}{p_1},$
$0 \leq \beta \leq \frac{p_2'(q_2 - \delta)}{p_1}$
$(3) : \alpha = \frac{Y_1 - p_2'(q_2 - \delta)}{p_1},$
$\beta = q_2 - \delta$</td></tr>
<tr><td>$\mathbf{III}.\ \mathbf{q_2} - \delta \geq \frac{\mathbf{Y_1}}{\mathbf{p_2'}}$</td><td colspan="2">$(1) : \alpha = q_1,\ \beta = \frac{Y_1 - p_1 q_1}{p_2'}$
$(2) : 0 \leq \alpha \leq q_1,\ \beta = \frac{Y_1 - p_1\alpha}{p_2'}$
$(3) : \alpha = 0,\ \beta = \frac{Y_1}{p_2'}$</td><td>$(1) : \alpha = \frac{Y_1}{p_1},\ \beta = 0$
$(2) : 0 \leq \alpha \leq \frac{Y_1}{p_1},$
$\beta = \frac{Y_1 - p_1\alpha}{p_2'}$
$(3) : \alpha = 0,\ \beta = \frac{Y_1}{p_2'}$</td></tr>
</table>

Shorthand notation used: (1) for $\mathsf{p}_1 < \mathsf{p}_2'$, (2) for $\mathsf{p}_1 = \mathsf{p}_2'$ and (3) for $\mathsf{p}_1 > \mathsf{p}_2'$

Table 1. Best-reply correspondence for consumer I.

With the aid of the best-reply correspondences we can compute the Nash equilibria for the game as solutions to a system of simultaneous equations. However, uniqueness is not guaranteed in this model and we therefore have to resort to additional rules for equilibrium selection in order to choose a single equilibrium. To this end we define the following supplementary selection rules (SR), which we deem logical from a practical viewpoint:

SR1 For a set of Nash equilibria yielding the same utility we select the one minimizing the expenditures made.

SR2 If more than one Nash equilibrium with the same utility can be obtained with the same (minimal) expenditure, then we select the one in which consumers receive the maximum amount possible in their own region in preference over the "foreign" consumer.

SR3 In the degenerate case when a price is equal to zero, we assume that the actual amount bought is equal to the quantity available in the respective region.

<table>
<tr><td></td><td colspan="2">$\mathbf{A} : \frac{\mathbf{Y_2}}{\mathbf{p_2}} > \mathbf{q_2}$</td><td>$\mathbf{B} : \frac{\mathbf{Y_2}}{\mathbf{p_2}} \le \mathbf{q_2}$</td></tr>
<tr><td>$\mathbf{I}.\ \mathbf{q_1} - \alpha \le \mathbf{0}$</td><td colspan="2">$0 \le \gamma \le \frac{Y_2 - p_2 q_2}{p_1'},\ q_2 \le \delta \le \frac{Y_2 - p_1' \gamma}{p_2}$</td><td>$\gamma = 0,\ \delta = \frac{Y_2}{p_2}$</td></tr>
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<tr><td>$q_1 - \alpha \le \gamma \le \frac{Y_2 - p_2 q_2}{p_1'}$
$q_2 \le \delta \le \frac{Y_2 - p_1' \gamma}{p_2}$</td><td>$(1) : \gamma = \frac{Y_2 - p_2 q_2}{p_1'},\ \delta = q_2$
$(2) : \frac{Y_2 - p_2 q_2}{p_1'} \le \gamma \le q_1 - \alpha,$
$\delta = \frac{Y_2 - p_1' \gamma}{p_2}$
$(3) : \gamma = q_1 - \alpha,$
$\delta = \frac{Y_2 - p_1' \gamma}{p_2}$</td><td>$(1) : \gamma = 0,\ \delta = \frac{Y_2}{p_2}$
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$(2) : 0 \le \gamma \le \frac{Y_2}{p_1'},$
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$(3) : \gamma = \frac{Y_2}{p_1'},\ \delta = 0$</td></tr>
</table>

Shorthand notation used: (1) for $\mathsf{p}_1' > \mathsf{p}_2$, (2) for $\mathsf{p}_1' = \mathsf{p}_2$ and (3) for $\mathsf{p}_1' < \mathsf{p}_2$

Table 2. Best-reply correspondence for consumer II.

4. Price dynamics. In order to see how prices evolve towards an equilibrium we represent graphically the set of financial resources (Y_1, Y_2) and partition the quadrant in an appropriate manner according to the type of Nash equilibrium obtained from Tables 1 and 2. A typical partition is presented in Figure 1.

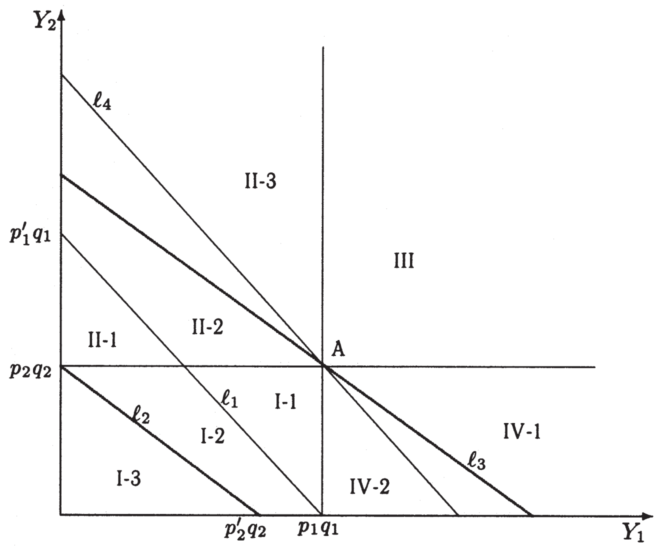

Fig. 1. Income space partition generated by the Nash equilibria for a fixed parameter set.

The lines $Y_i = p_i q_i$ divide the quadrant into four zones, denoted by roman numerals in the figure. For simplicity we posit $p_1 q_1 > p_2 q_2$ for the initial setup, the other case being fully symmetric. The restrictions coming from the columns of Tables and define the following lines:

$$\ell_1: \ p_1 q_1 = Y_1 + (p_1/p_1')Y_2,$$

$$\ell_2: \ p_2 q_2 = (p_2/p_2')Y_1 + Y_2,$$

$$\ell_3: \ p_1 q_1 = Y_1 - p_2'(q_2 - Y_2/p_2) \text{ and}$$

$$\ell_4: \ p_2 q_2 = Y_2 - p_1'(q_1 - Y_1/p_1).$$

It is obvious that ℓ_1 and ℓ_4 are parallel, as are ℓ_2 and ℓ_3. The segment of ℓ_4 crossing into zone IV does not affect the subsequent calculations, therefore we divide zone IV into two sub-zones (IV-1 and IV-2) determined by ℓ_3. Similarly, for zone II the corresponding segment of the line ℓ_3 is irrelevant for the analysis and the zone is divided into three sub-zones determined by the segments of lines ℓ_1 and ℓ_4 passing through it. The partitioning of zone I is based on the same principle.

The approach we adopt in the calculations is as follows. For each of the sub-zones we obtain the form of the Nash equilibrium, using Tables 1 and 2, solving the system and applying the supplementary rules, if necessary. We then use the outcome to check whether there are unutilized quantities or incomes, implying that a price adjustment is needed. If no adjustment is necessary, we conclude that prices are in equilibrium and stop, otherwise we change the prices according to either (3) or (4) and repeat the procedure. It is important to note that a pair (Y_1, Y_2) is fixed from the start and in the course of the iterations it can fall into a different zone only because price changes shift the lines in the income space. In other words, the partition changes from one iteration to the next and thus different sub-zones may cover the fixed point (Y_1, Y_2).

The typical results we obtain are illustrated below for three of the zones. For example, in zone III the Nash equilibrium is of the form $(q_1, 0, 0, q_2)$. If it turns out that $Y_i = p_i q_i, \ i = 1, 2$, this means that both the quantities and the financial resources have been depleted and the point A is an equilibrium. If $Y_i > p_i q_i$ for some $i = 1, 2$, then (4) gives $p_{i,t+1} = Y_i/q_i$ and we arrive at the new point A, where no further corrections are needed.

For sub-zone II-3 we obtain the Nash equilibrium $(Y_1/p_1, 0, q_1 - Y_1/p_1, q_2)$. Thus q_1, Y_1 and q_2 are used up and therefore p_1 is left unchanged. Here $Y_2 > p_2 q_2$ and $Y_2^{res} = Y_2 - p_1'(q_1 - Y_1/p_1) \geq p_2 q_2$. If $Y_2^{res} = p_2 q_2$ the price p_2 is also left unchanged and points on the corresponding segment of ℓ_4 are equilibria. If $Y_2^{res} > p_2 q_2$ the price p_2 increases to Y_2^{res}/q_2 and we end up on the new ℓ_4 segment, reaching an equilibrium.

Turning to sub-zone II-2, we first look at the case where $p_1' \geq p_2$. In this case the Nash equilibrium is $(Y_1/p_1, 0, (Y_2 - p_2 q - 2)/p_1', q_2)$. If $p_1' < p_2$, the Nash equilibrium is of the form $(Y_1/p_1, 0, q_1 - Y_1/p_1, (Y_2 - p_1'(q_1 - Y_1/p_1))/p_2)$. For the latter case and assuming $q_2^{cons} > 0$, we have p_1 unchanged and p_2 decreasing to $(Y_2 - p_1'(q_1 - Y_1/p_1))/q_2$, which means that the system jumps to the new ℓ_4 line. If $q_2^{cons} = 0$, p_2 decreases to zero and the degenerate case obtains. Here the Nash equilibrium is $(Y_1/p_1, 0, Y_2/p_1', 0)$. When $p_1' \geq p_2$, p_2 is left unchanged while p_1 decreases. In this case, if $Y_2 = p_2 q_2$ the equilibrium will take the form $(Y_1/p_1, 0, 0, q_2)$ and after one correction of p_1 we get to the new point A. If $Y_2 > p_2 q_2$, it turns out that the new ℓ_4 line rotates to a position above the point (Y_1, Y_2), i.e. gets steeper. Then if $p_1' = p_2$, we adjust to $p_1' < p_2$, which was described above. If $p_1' > p_2$, the point (Y_1, Y_2) is above the line $\bar{\bar{\ell}}: \ y + (p_1/p_1')z = (p_2 - \rho)q_1 + (p_1/p_1')p_2 q_2$.

Thus the points in II-2 above ℓ_1 and below $\bar{\bar{\ell}}$ after the adjustment in p_1 will be in the case $p'_{1,t+1} < p_{2,t+1}$ (provided they remain in II-2 at all), and after $p_{2,t+1}$ is adjusted an equilibrium will be reached. It can also be shown that (Y_1, Y_2) can fall below the new ℓ_1 line (i.e. fall into sub-zone II-1) under certain conditions. Finally, for (Y_1, Y_2) between $\bar{\bar{\ell}}$ and ℓ_4, there will be an infinite price convergence process under which the line ℓ_4 will rotate to a limit line containing the point (Y_1, Y_2).

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ДИНАМИКА НА ЦЕНИТЕ В МОДЕЛ НА СТРАТЕГИЧЕСКО ВЗАИМОДЕЙСТВИЕ МЕЖДУ ДВА РЕГИОНА

Йордан В. Йорданов, Стоян В. Стоянов, Андрей Ал Василев

Статията резработва стратегически модел на търговията между два региона, в които в зависимост от отношението между производство, транспортни разходи и финансови ресурси се изследва процесът на изменение на цените при сходимостта им към равновесие. Изведени са условия за връзките между производството и финансовите ресурси, при които се получават различни видове равновесия по Наш. Също така са изведени траекториите за цените, получени в процеса на достигане на равновесие.