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CLIQUES WITH MAXIMAL DEGREE IN GRAPHS*

N. Khadzhiivanov

The class of the p-ordered graphs is introduced (cf. Definition). It is proved that in any p-ordered graph there exists a p-clique such that the arithmetic mean of its vertex degrees is at least equal to the arithmetic mean of all vertex degrees in the graph.

Let G be a graph, $d\,(v)$ – the degree of the vertex v in G. Given the vertex set M, we denote $d\,(\mathrm{M}) = \frac{1}{|M|}\sum_{v\in M} d(v)$ and call it the degree of M.

In [2] we proved the following

Theorem 1. *Let G be a nonregular graph and V be the set of all vertices of G. If the natural number p satisfies the inequality $p < |V|$, then there exists a p-vertex subset M of V with $d(M) > d(V)$.*

For $p \geq 3$ it is impossible to strenghten this proposition by the additional statement that M is a clique (i.e. any two vertices are adjacent). R. Faudree proved in [1] that if G is a graph with n vertices and at least $\frac{p-1}{2p}n^2$ edges, then there exists a p-clique K such that $d(K) \geq d(V)$. In this article, we determine another condition that is sufficient for the existence of a p-clique K with $d(K) \geq d(V)$.

Definition. *We call p-degree of a vertex v and denote it by $d_p(v)$ the number of all p-cliques containing v. The graph G is said to be* p-ordered*, if it contains a p-clique and the inequality*

$$(d(u) - d(v))(d_p(u) - d_p(v)) \geq 0$$

holds for each pair of vertices u, v.

Every graph is 2-ordered. Any regular or any p-regular graph is p-ordered. It is not difficult to prove that any complete s-partite graph is p-ordered for all p.

When G is a p-ordered graph that is neither regular nor p-regular, Theorem 1 may be precised as:

Theorem 2. *Let G with a vertex set V be p-ordered graph that is neither regular nor p-regular. Then the inequality $d(K) > d(V)$ holds for some p-clique K.*

Theorem 2 may be completed in the following way:

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Theorem 3. *Let G be a p-ordered graph and V be its vertex set. Then the inequality $d(K) \geq d(V)$ holds for some p-clique K.*

Proof of Theorems 2 and 3. Let G be a graph. We denote by $C_p(G)$ the set of all p-cliques in G. In our proof we use the equality

(1) $$\sum_{v\in V} d_p(v) = p|C_p(G)|$$

This is true, since each p-clique K in G contributes 1 only to those terms $d(v)$, for which $v \in K$, i.e. p times.

We need also the equality

(2) $$\sum_{K\in C_p(G)} d(K) = \frac{1}{p}\sum_{v\in V} d(v).d_p(v).$$

It holds as in the lefthand sum each degree $d(v)$ occurs as many times, as is the number of the p-cliques that contain v, i.e. $d_p(v)$ times.

Let us recall the famous inequality of P. Tchebishev (see [3], p. 43):

If the real numbers $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ satysfy the inequalities

(3) $$(x_i - x_j)(y_i - y_j) \geq 0, \qquad i, j \in \{1, 2, \ldots, n\},$$

then

(4) $$\frac{x_1 + \cdots + x_n}{n} \cdot \frac{y_1 + \cdots + y_n}{n} \geq \frac{x_1y_1 + \cdots + x_ny_n}{n}.$$

Furthermore, the equality in (4) holds if and only if $x_1 = \cdots = x_n$ or $y_1 = \cdots = y_n$.

Assume now that G is a p-ordered graph with vertex set $V = \{v_1, \ldots, v_n\}$. Let $x_i = d(v_i)$, $y_i = d_p(v_i)$. Then the inequalities (3) hold and, therefore, according to (4), we obtain

(5) $$\sum_{i=1}^{n} d(v_i).\sum_{i=1}^{n} d_p(v_i) \leq n\sum_{i=1}^{n} d(v_i)d_p(v_i).$$

Then (see (1), (2) and (5))

(6) $$d(V) \leq \frac{\sum\limits_{K\in C_p(G)} d(K)}{|C_p(G)|}.$$

The equality holds only when G is regular or p-regular.

With the aid of (6) it is easy to finish the proof.

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Faculty of Mathematics and Informatics
"St. Kl. Ohridski University" of Sofia
5, James Bourchier Blvd.
1164 Sofia, Bulgaria
e-mail: hadzhiivanov@hotmail.com

КЛИКИ С МАКСИМАЛНА СТЕПЕН В ГРАФИ

Николай Г. Хаджииванов

Нека G е граф и V е множеството от върховете му. За произволно множество M от върхове на графа под *степен на* M (означение $d(M)$) разбираме средното аритметично на степените на върховете на M.
В произволен нерегулярен n-върхов граф при $p < n$ съществува p-върхово подмножество M на V, за което $d(M) > d(V)$ (Теорема 1). Но при $p \geq 3$ не е сигурно, че може да се намери p-клика M, която удовлетворява това неравенство. Това обаче може да се направи за един клас графи. Под p-степен на върха v (означение $d_p(v)$) разбираме броя на p-кликите на графа, които съдържат v. Наричаме *p-нареден граф* този, който съдържа поне една p-клика и неравенството $(d(u)-d(v)).(d_p(u)-d_p(v)) \geq 0$ е в сила за всяка двойка върхове u, v. В произволен p-нареден граф може да се намери p-клика M, за която $d(M) \geq d(V)$. (Теорема 3). При това неравенството е дори строго, ако графът не е нито регулярен, нито p-регулярен (Теорема 2).