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ON THE NUMBER OF DISCRETE FUNCTIONS WITH A GIVEN C-SPECTRUM OR WITH A GIVEN SPECTRUM*

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In this paper, results of [5] are generalized. Let M be the set of variables of a discrete function. Range or the spectrum of M with respect to the function is continuation of the research connected with the essential set of variables for the function (see [1] and [4]) and can be used as a measure of essentiality.

1. Introduction. Let K, K_1, K_2, ..., K_n be finite, non-empty sets, and K={ 0, 1,..., k-1}, $|K| = k \geq 2$, K_i={ 0, 1,..., k_i-1}, $|K_i| = k_i \geq 1$, i=1, 2,..., n, where the cardinality of the set K is denoted by $|K|$.

Let us set $X = K_1 \times K_2 \times \cdots \times K_n = \{(c_1, c_2, \ldots, c_n)|\ c_i \in K_i,\ i = 1, \ldots, n\}$ and let F_n^k be the set of all functions of n variables defined in the set X and having values in the set K.

In the special case $K_1 = K_2 = \cdots = K_n = K = \{0, 1, \ldots, k-1\}$, we obtain the set of all functions of the k-valued logic, which is denoted by P_n^k.

Definition 1.1 [1] *The number of all different values of the function f is called the **range** of f.*

We will denote the ***range*** of the function f by $\boldsymbol{Rng}(f)$ and by X_f we will denote the set of variables of the function $f(x_1, x_2, \ldots, x_n)$, i.e. $X_f = \{x_1, x_2, \ldots, x_n\}$.

We will denote [1] by λ the number of all possible sets of constants for the variables of the functions of F_n^k, where

(1) $$\lambda = |X| = k_1 k_2 \ldots k_n,$$

and by λ_M the number of all possible sets of constants for the variables of the set $M = \{x_{j_1}, x_{j_2}, \ldots, x_{j_m}\}$, $M \subseteq X_f$, where

(2) $$\lambda_M = k_{j_1} k_{j_2} \ldots k_{j_m}.$$

Definition 1.2 [3] *The function h is called a subfunction of the function f with respect to R, $R \subseteq X_f$, if h is obtained from f by substitution of the variables of the set R with constants, and this is denoted by $h \overset{R}{\prec} f$.*

Let M, $M \subseteq X_f$, be a set of variables of the function $f \in F_n^k$ and G be the set of all subfunctions of f with respect to $X_f \backslash M$, i.e. $G = G(M, f) = \{g : g \overset{X_f \backslash M}{\prec} f\}$.

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Definition 1.3 [1] *If $g \in G$, then the* ***range*** *of the subfunction g is called* ***range*** *of the set M for the function f with respect to g.*

By $\boldsymbol{Rng}(M, f; g)$ we denote the ***range*** of the set M for f with respect to g, and

$$\boldsymbol{Rng}(M, f; g) = \boldsymbol{Rng}(g).$$

Definition 1.4 [1] *The set $\boldsymbol{Spr}(M, f) = \bigcup_{g \in G} \boldsymbol{Rng}(M, f; g) = \bigcup_{g \in G} \boldsymbol{Rng}(g)$ is called* ***spectrum*** *of the set M for the function f.*

Definition 1.5 [1] *The number* $\max \boldsymbol{Spr}(M, f)$ *is called the* ***range*** *of M for the function f.*

The range of M for the function f will be denoted by $\boldsymbol{Rng}(M, f)$, and

$$\boldsymbol{Rng}(M, f) = \max \boldsymbol{Spr}(M, f) = \max(\bigcup_{g \in G} \boldsymbol{Rng}(M, f; g)) = \max(\bigcup_{g \in G} \boldsymbol{Rng}(g))$$

Definition 1.6 [5] *The set $\{1^{p_1}, 2^{p_2}, \dots, k^{p_k}\}$ is called* ***C-spectrum*** *of M for f, where p_t, $p_t \geq 0$, $t = 1, \dots, k$, is the number of the different sets of values for the variables of the set $X_f \backslash M$, by which from f we obtain subfunctions with a range equal to t, and*

$$p_1 + p_2 + \dots + p_k = \frac{\lambda}{\lambda_M} \text{ for } f \in F_n^k, \text{ and } p_1 + p_2 + \dots + p_k = k^{n-|M|} \text{ for } f \in P_n^k.$$

The C-spectrum of M for f is denoted by $\boldsymbol{C}$-**Spr** (M, f), where $\boldsymbol{C}$-$\boldsymbol{Spr}(M, f) = \{1^{p1}, \dots, k^{pk}\}$.

2. Results. If M is a non-empty set of variables of a function of F_n^k and $M = \{x_{j1}, x_{j2}, \dots, x_{jm}\}$, then let $X_M = K_{j1} \times K_{j2} \times \dots \times K_{jm}$ and $F_M^k = \{h : X_M \to K\}$ be the set of all functions of the variables of the set M defined in the set X_M and having values in the set K.

For $\lambda_M \geq q$ and $q \in \{1, 2, \dots, k\}$, let us denote [2] by $\mu_M^k(q)$ the number of functions of F_M^k with a range equal to q, where for $\mu_M^k(q)$ we have

$$(3) \qquad \mu_M^k(q) = C_k^q \cdot \sum_{\substack{r_1 + r_2 + \dots + r_q = \lambda_M \\ r_i \geq 1, \ i = 1, 2, \dots, q}} \frac{\lambda_M!}{r_1! r_2! \dots r_q!} = C_k^q \cdot \sum_{j=1}^{q} (-1)^{q-j} C_q^j j^{\lambda_M}.$$

Theorem 2.1. *If $\emptyset \neq M = \{x_{j_1}, x_{j_2}, \dots, x_{j_m}\}$, $M \subset X_f$, then the number of functions $f \in F_n^k$ for which $\boldsymbol{C}$-$\boldsymbol{Spr}(M, f) = \{1^{p_1}, 2^{p_2}, \dots, k^{p_k}\}$ is equal to*

$$\frac{(\lambda/\lambda_M)!}{p_1! p_2! \dots p_k!} \alpha_1^{p_1} \cdot \alpha_2^{p_2} \dots \alpha_k^{p_k}, \quad \text{where } \alpha_t = \mu_M^k(t), \ t = 1, \dots, k.$$

Proof. Let $M = \{x_{j_1}, x_{j_2}, \dots, x_{j_m}\}$ and that $X_f \backslash M = \{x_{j_{m+1}}, \dots, x_{j_n}\}$.

Let us denote the number of the different sets of constants for the variables of $X_f \backslash M$ by s. Under the terms of formula (2) we have

$$\lambda_{X_f \backslash M} = s = k_{j_{m+1}} \cdot k_{j_{m+2}} \dots k_{j_n} = (k_1 k_2 \dots k_n)/(k_{j_1} k_{j_2} \dots k_{j_m}) = \lambda/\lambda_M.$$

Let all possible sets of constants for the variables of $X_f \backslash M$ be $(c^i_{m+1}, \ldots, c^i_n)$, $i = 1, 2, \ldots, s$. If the function f is from the ones we look for and

$$g_i = f(x_{j_{m+1}} = c^i_{m+1}, \ldots, x_{j_n} = c^i_n), \quad i = 1, 2, \ldots, s, \tag{4}$$

then it is obvious that

$$g_i \overset{X_f \backslash M}{\prec} f, \quad g_i \in G \text{ and } g_i \in F^k_n, \quad i = 1, 2, \ldots, s.$$

In Table 2.1 the function f is presented in tabular form, in accordance with the equations (4). From Table 2.1 we can see that the function f consists of s parts, and the parts are the subfunctions g_i, $i = 1, 2, \ldots, s$.

The function f depends on the subfunctions g_i, $i = 1, 2, \ldots, s$, and on the place they take.

Let us associate the number t, $t = 1, \ldots, k$, to the sets of constants for the variables of $X_f \backslash M$, to which subfunctions with a range equal to t correspond.

From $\textbf{\textit{C-Spr}}\,(M, f) = \{1^{p_1}, 2^{p_2}, \ldots, k^{p_k}\}$ it follows that we have p_t, $t = 1, \ldots, k$, sets of constants for the variables of $X_f \backslash M$, to which we have associated the number t (or we have an s-dimensional vector, in which the number t occurs p_t times, $t = 1, \ldots, k$).

The number of the different mappings, in which we associate the number t to p_t, $t = 1, \ldots, k$, sets of constants for the variables of $X_f \backslash M$ is equal to

$$\frac{(p_1 + p_2 + \cdots + p_k)!}{p_1! p_2! \ldots p_k!} = \frac{(\lambda/\lambda_M)!}{p_1! p_2! ... p_k!}$$

Since we associate to every number t subfunction of F^k_M with a range equal to t, which we can choose among $\alpha_t = \mu^k_M(t)$ such subfunctions (under the terms of (3)), $t = 1, \ldots, k$, and having in mind that a subfunction with a range equal to t must be associated to p_t sets of constants for the variables of $X_f \backslash M$, then finally for the number of the functions $f \in F^k_n$, for which $\textbf{\textit{C-Spr}}\,(M, f) = \{1^{p_1}, 2^{p_2}, \ldots, k^{p_k}\}$, we get

$$\frac{(\lambda/\lambda_M)!}{p_1! p_2! ... p_k!} \alpha_1^{p_1} \cdot \alpha_2^{p_2} \ldots \alpha_k^{p_k}, \tag{5}$$

where $\alpha_t = \mu^k_M(t)$, $t = 1, \ldots, k$ and $p_1 + p_2 + \cdots + p_k = \lambda/\lambda_M$.

If $p_i = 0$, $i \in \{1, 2, \ldots, k\}$ then $(p_i)! = 1$, $\alpha_i^{p_i} = \alpha_i^0 = 1$ and it follows that the result from formula (5) does not depend on p_i.

Corollary 2.1. *If $\emptyset \neq M = \{x_{j_1}, x_{j_2}, \ldots, x_{j_m}\}$, $M \subset X_f$, then the number of functions $f \in F^k_n$, for which $\textbf{\textit{C-Spr}}\,(M, f) = \{q_1^{v_1}, q_2^{v_2}, \ldots, q_s^{v_s}\}$, $v_i > 0$, $q_i \leq k$, $i = 1, 2, \ldots, s$, is equal to*

$$\frac{(\lambda/\lambda_M)!}{v_1! v_2! \cdots v_s!} \cdot \rho_1^{v_1} \cdot \rho_2^{v_2} \ldots \rho_s^{v_s}, \quad \textit{where} \ \ \rho_t = \mu^k_M(q_t), \quad t = 1, 2, \ldots, s \ \ \textit{and} \ \ \sum_{i=1}^{s} v_i = \lambda/\lambda_M.$$

For the functions of P^k_n the numbers λ_M and $\mu^k_M(q)$ do not depend on the variables in the set M, but they depend on the cardinality of the set M only.

If $|M| = m$ for the functions of P^k_n we have:

$$\lambda_m = \lambda_M = \lambda_{|M|} = k^m, \quad \lambda = k^n, \quad \lambda/\lambda_M = k^{n-|M|} = k^{n-m} \ \text{ and}$$

$$\mu^k_m(q) = \mu^k_M(q) = C^q_k \cdot \sum_{\substack{r_1 + r_2 + \cdots + r_q = k^m \\ r_i \geq 1, \ i = 1, 2, \ldots, q}} \frac{k^m!}{r_1! r_2! \ldots r_q!} = C^q_k \cdot \sum_{j=1}^{q} (-1)^{q-j} C^j_q j^{k^m}.$$

x_{j_1}	x_{j_2}	$\cdots$	x_{j_m}	$x_{j_{m+1}}$	$\cdots$	x_{j_n}	$f(x_{j_1}, x_{j_2}, \ldots, x_{j_m}, x_{j_{m+1}}, \ldots, x_{j_n})$	g_i
0	0	$\cdots$	0	c^1_{m+1}	$\cdots$	c^1_n	$f(0,0,\ldots,0,c^1_{m+1},\ldots,c^1_n)$=$g_1(0,0,\ldots,0)$	
0	0	$\cdots$	1	c^1_{m+1}	$\cdots$	c^1_n	$f(0,0,\ldots,1,c^1_{m+1},\ldots,c^1_n)$=$g_1(0,0,\ldots,1)$	
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots\cdots\cdots\cdots\cdots\cdots$	g_1
$k_{j_1}-1$	$k_{j_2}-1$	$\cdots$	$k_{j_m}-1$	c^1_{m+1}	$\cdots$	c^1_n	$f(k_{j_1}-1,k_{j_2}-1,\ldots,k_{j_m}-1,c^1_{m+1},\ldots,c^1_n)=$ $= g_1(k_{j_1}-1,k_{j_2}-1,\ldots,k_{j_m}-1)$	
0	0	$\cdots$	0	c^2_{m+1}	$\cdots$	c^2_n	$f(0,0,\ldots,0,c^2_{m+1},\ldots,c^2_n) = g_2(0,0,\ldots,0)$	
0	0	$\cdots$	1	c^2_{m+1}	$\cdots$	c^2_n	$f(0,0,\ldots,1,c^2_{m+1},\ldots,c^2_n)$=$g_2(0,0,\ldots,1)$	
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots\cdots\cdots\cdots\cdots\cdots$	g_2
$k_{j_1}-1$	$k_{j_2}-1$	$\cdots$	$k_{jm}-1$	c^2_{m+1}	$\cdots$	c^2_n	$f(k_{j_1}-1,k_{j_2}-1,\ldots,k_{j_m}-1,\ c^2_{m+1},\ldots,c^2_n)=$ $= g_2(k_{j_1}-1,k_{j_2}-1,\ldots,k_{j_m}-1)$	
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots\cdots\cdots\cdots\cdots\cdots$	$\cdots$
0	0	$\ldots$	0	c^s_{m+1}	$\ldots$	c^s_n	$f(0,0,\ldots,0,c^s_{m+1},\ldots,c^s_n)$=$g_s(0,0,\ldots,0)$	
0	0	$\ldots$	1	c^s_{m+1}	$\ldots$	c^s_n	$f(0,0,\ldots,1,c^s_{m+1},\ldots,c^s_n)$=$g_s(0,0,\ldots,1)$	
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots\cdots\cdots\cdots\cdots\cdots$	g_s
$k_{j_1}-1$	$k_{j_2}-1$	$\ldots$	$k_{j_m}-1$	c^s_{m+1}	$\ldots$	c^s_n	$f(k_{j_1}-1,k_{j_2}-1,\ldots,k_{j_m}-1,\ c^s_{m+1},\ldots,c^s_n)=$ $= g_s(k_{j_1}-1,k_{j_2}-1,\ldots,k_{j_m}-1)$	

Table 2.1

Corollary 2.2. *If $\emptyset \neq M \subset X_f$, $|M| = m$, then the number of functions $f \in P^k_n$, for which **C-Spr**$(M,f) = \{q_1^{v_1}, q_2^{v_2}, \ldots, q_s^{v_s}\}$, $v_i > 0$, $q_i \in \{1,\ldots,k\}$, $i = 1,2,\ldots,s$ is equal to*

$$\frac{k^{n-m}!}{v_1!v_2!\ldots v_s!}\cdot\rho_1^{v_1}\cdot\rho_2^{v_2}\ldots\rho_s^{v_s},$$

where $\rho_t = \mu^k_m(q_t)$, $t = 1,2,\ldots,s$ and $\sum\limits_{i=1}^{s} v_i = k^{n-m}$.

Theorem 2.2. *If $\emptyset \neq M = \{x_{j_1}, x_{j_2}, \ldots, x_{j_m}\}$, $M \subseteq X_f$, then the number of functions $f \in F^k_n$, for which **Spr**$(M,f) = \{q_1, q_2, \ldots, q_s\}$, $s \leq k$, $q_i \leq k$, $i = 1,2,\ldots,s$ is equal to*

$$\sum_{\substack{r_1+r_2+\cdots+r_s=\lambda/\lambda_M \\ r_i\geq 1,\ i=1,2,\ldots,s}} \frac{(\lambda/\lambda_M)!}{r_1!r_2!\ldots r_s!}\cdot\rho_1^{r_1}\cdot\rho_2^{r_2}\ldots\rho_s^{r_s}, \quad \textit{where} \;\; \rho_I = \mu^k_M(q_i), \quad i = 1,2,\ldots,s.$$

Proof. Let us denote by r_i the number of the sets of constants for the variables of $X_f \backslash M$, by which from f we obtain subfunctions with a range equal to q_i, $i = 1,2,\ldots,s$.

In this case for the set M and the function f we have

$$\textbf{\textit{C-Spr}}\,(M,f) = \{q_1^{r_1}, q_2^{r_2}, \ldots, q_s^{r_s}\}.$$

Taking into consideration that every function f, for which
$\boldsymbol{C}-\boldsymbol{Spr}\,(M,f)=\{q_1^{r_1},q_2^{r_2},\dots,q_s^{r_s}\}$, where $r_1+r_2+\cdots+r_s=\lambda/\lambda_M$, $r_i\geq 1$, $i=1,2,\dots,s$, has $\boldsymbol{Spr}\,(M,f)=\{q_1,q_2,\dots,q_s\}$, and applying Theorem 2.1, we get the proof of the theorem.

Corollary 2.3. [5] *If $\emptyset\neq M\subseteq X_f, |M|=m$, then the number of functions $f\in P_n^k$, for which $\boldsymbol{Spr}\,(M,f)=\{q_1,q_2,\dots,q_s\}$, $q_i\leq k$, $i=1,2,\dots,s$, $s\leq k$, is equal to*

$$\sum_{\substack{v_1+v_2+\cdots+v_s=k^{n-m}\\ v_i\geq 1,\ i=1,2,\dots,s}}\frac{k^{n-m}!}{v_1!v_2!\dots v_s!}\rho_1^{v_1}\rho_2^{v_2}\dots\rho_s^{v_s},$$

where

$$\rho_t=\mu_m^k(q_t),\quad t=1,2,\dots,s\quad \textit{and}\quad \sum_{i=1}^{s}v_i=k^{n-m}.$$

The proof of Theorem 2.1 can be used for the "construction", i.e. the tabular presentation of functions with a given *C-spectrum.*

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ВЪРХУ БРОЯ НА ДИСКРЕТНИ ФУНКЦИИ С ДАДЕН *C*-СПЕКТЪР ИЛИ С ДАДЕН СПЕКТЪР

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В настоящата статия се обобщават резултати от [5]. Нека M е множество от променливи на дискретна функция. Рангът или спектърът на M относно функцията, освен че са продължение на изследванията свързани със съществено множество от променливи на функция (виж [1] и [4]), могат да се използват и като мярка за същественост.