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SOME SUBMANIFOLDS OF CODIMENSION TWO OF ALMOST COMPLEX MANIFOLDS WITH B-METRIC

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Submanifolds of codimension two of almost complex manifolds with B-metric in the case when the normal section is totally real are considered. These submanifolds are not holomorphic. An almost complex structure and B-metric on the submanifolds are defined. A class of some submanifolds of a Kaehler manifold with B-metric is found.

1. Introduction. Let (M,J,g) be a $2n-$dimensional almost complex manifold with B-metric, i.e. J is the almost complex structure and g is the metric on M such that:

$$J^2X=-X;\quad g(JX,JY)=-g(X,Y).$$

for all vector fields X,Y on M. The associated with g metric $\tilde{g}$ on the manifold is given by $\tilde{g}(X,Y)=g(JX,Y)$. Both metrics g and $\tilde{g}$ are indefinite of signature (n,n).

Let ∇ be the Levi-Civita connection of the metric g. The tensor field F of type $(0,3)$ on M is defined by $F(X,Y,Z)=g((\nabla_X J)Y,Z)$. This tensor has the following symmetries:

$$F(X,Y,Z)=F(X,Z,Y);\quad F(X,Y,Z)=F(X,JY,JZ).$$

The Lie form θ associated with the tensor F is defined by $\theta(x)=-\dfrac{1}{n}g^{ij}F(e_i,e_j,Jx)$, where $x\in T_pM$, $\{e_i\}(i=1,2,\ldots,2n)$ is an arbitrary basis of T_pM and (g^{ij}) is the inverse matrix of (g_{ij}).

A classification of the almost complex manifolds with B-metric with respect to the tensor F is given in [2]. The class of the Kaehler manifolds with B-metric is defined by the condition $F(X,Y,Z)=0$.

Let R be the curvature tensor field of type $(1,3)$ of the Levi-Civita connection ∇ of g

$$R(X,Y,Z)=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z.$$

The corresponding to R tensor field of type $(0,4)$ is given by

$$R(X,Y,Z,W)=g(R(X,Y,Z),W)$$

Let α be a $2-$dimensional section in T_pM. A classification of the sections in T_pM is given in [1]. Let us recall: a section α is said to be nondegenerate, weakly isotropic or strongly isotropic if the rank of the restriction of the metric g $(\tilde{g})$ on α is $2,1$, or 0 respectively. A section α is said to be holomorphic if $J\alpha=\alpha$ and totally real – if $J\alpha\perp\alpha$. A section α is of pure or hybrid type if the restriction of g $(\tilde{g})$ on α has a signature $(2,0),(0,2)$ or $(1,1)$ respectively.

2. Submanifolds of codimension 2 of an almost complex manifold with B-metric. Let $(\bar{M}, J, g)$ be an almost complex manifold with B-metric and let M be a submanifold of $\bar{M}$. By T_pM and $(T_pM)^\perp$ are denoted the tangent space and the normal space of M at $p \in M$ respectively. Also, by TM and $(TM)^\perp$ are denoted the tangent bundle and the normal bundle of M, respectively. In [3] definitions for holomorphic, totally real and **CR** submanifold of a Hermitian manifold are given. Similar definitions we apply to submanifolds of an almost complex manifold with B-metric:

- If $J(T_pM) \subset T_pM$ for each $p \in M$, then M is called a holomorphic (or invariant) submanifold of $\bar{M}$;
- If $J(T_pM) \subset (T_pM)^\perp$ for each $p \in M$, then M is called a totally real (or anti-invariant) submanifold of $\bar{M}$;
- A submanifold M of $\bar{M}$ is called a **CR** submanifold of $\bar{M}$ if there exists a differentiable distribution $D : p \to D_p$ on M satisfying the two conditions:

  (**i**) D is invariant, that is, $JD_p = D_p$ for each $p \in M$;

  (**ii**) the complementary orthogonal distribution $D^\perp : p \to D_p^\perp \subset T_pM$ of D is anti-invariant, that is, $JD_p^\perp \subset (T_pM)^\perp$ for each $p \in M$.

In [4] holomorphic submanifolds of almost complex manifolds with B-metric are studied. In this section we consider submanifolds of codimension 2 of a Kaehler manifold with B-metric when the normal section $\alpha = (T_pM)^\perp$ is totally real, i.e. $J\alpha \subset T_pM$.

Let $(\bar{M}, J, g)$ be a $(2n+2)-$dimensional almost complex manifold with B-metric g and let M be a submanifold of codimension 2 of $\bar{M}$. Let $\alpha = \{N_1, N_2\}$ be a normal section defined globally over the submanifold M such that:

$$-g(N_1, N_1) = g(N_2, N_2) = 1, \quad g(N_1, N_2) = 0. \tag{2.1}$$

We consider the following decomposition for $JN_1, JN_2, JX\ (X \in TM)$ with respect to $\{N_1, N_2\}$ and TM:

$$JN_1 = \xi_1, \tag{2.2}$$

$$JN_2 = \xi_2, \tag{2.3}$$

$$JX = \varphi X + \eta^1(X)N_1 + \eta^2(X)N_2 \quad X \in TM, \tag{2.4}$$

where φ denotes a tensor field of type $(1,1)$ on M; $\xi_1, \xi_2 \in TM$ and η^1, η^2 are 1-forms on M. We denote the restriction of g on M by the same letter. Taking into account (2.1), (2.2) and (2.3) we have that the normal section $\alpha = \{N_1, N_2\}$ is a nondegenerate totally real section of hybrid type. These sections exist by $\dim \bar{M} \geq 4$ [1]. From (2.1), (2.2) and (2.3) we obtain

$$g(\xi_1, \xi_1) = 1, \; g(\xi_2, \xi_2) = -1, \; g(\xi_1, \xi_2) = 0. \tag{2.5}$$

Using the equalities (2.1)÷(2.4) we find

$$\eta^1(X) = -g(X, \xi_1), \quad \eta^2(X) = g(X, \xi_2). \tag{2.6}$$

Proposition 2.1. *The submanifold M of codimension 2 of $(\bar{M}, J, g)$ in the case when the normal section $\alpha = \{N_1, N_2\}$ is nondegenerate totally real of hybrid type is not holomorphic.*

Proof. If we assume that M is holomorphic, then $g(JX, N_1) = g(JX, N_2) = 0$ for each $X \in TM$. Hence, for $\xi_1, \xi_2 \in TM$ we have $g(J\xi_i, N_i) = g(\xi_i, JN_i) = g(\xi_i, \xi_i) = 0$ $(i = 1, 2)$, which is in contradiction to (2.5).

Let $H_pM = J(T_pM) \bigcap T_pM$ be the maximal holomorphic tangent space to M at p. It is an even-dimensional subspace of T_pM, since $J^2_{|H_pM} = -id$. Moreover, from (2.2), (2.3) we have $J(\xi_1)_p, J(\xi_2)_p \in (T_pM)^\perp$ and for any $x \in H_pM$ $g(x, (\xi_i)_p) = g(x, (JN_i)_p) = g(Jx, (N_i)_p) = 0$ $(i = 1, 2)$. So, there are two distributions on M $D : p \to H_pM$, $D^\perp : p \to \{(\xi_1)_p, (\xi_2)_p\}$ satisfying $JD_p = D_p$, $J(D^\perp_p) \subset (T_pM)^\perp$ and $D_p \perp \{(\xi_1)_p, (\xi_2)_p\}$ for each $p \in M$. Then $T_pM = H_pM \oplus \operatorname{span}\{(\xi_1)_p, (\xi_2)_p\}$, where $\dim(H_pM) = 2n - 2$ and the submanifold M of $(\bar{M}, J, g)$ is a **CR** submanifold.

From the last notes and Proposition 2.1 it follows that we can consider a submanifold M of codimension 2 of $(\bar{M}, J, g)$ such that $J(T_pM) \subset T_pM \oplus (T_pM)^\perp$ for each $p \in M$, i.e. in (2.4) $(\eta^1(X), \eta^2(X)) \neq (0, 0)$.

Applying J to (2.2), (2.3), (2.4) and comparing the tangential parts and the normal parts to M, we have

$$\varphi\xi_1 = \varphi\xi_2 = 0, \tag{2.7}$$

$$\eta^1(\varphi X) = \eta^2(\varphi X) = 0, \tag{2.8}$$

$$\varphi^2 X = -X - \eta^1(X)\xi_1 - \eta^2(X)\xi_2. \tag{2.9}$$

Using $g(JX, JY) = -g(X, Y)$ and (2.4) we obtain

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta^1(X)\eta^1(Y) - \eta^2(X)\eta^2(Y), \quad X, Y \in TM. \tag{2.10}$$

Now we define a tensor field J' of type $(1, 1)$ on M by

$$J'X = \varphi X + \eta^2(X)\xi_1 - \eta^1(X)\xi_2, \quad X \in TM. \tag{2.11}$$

Taking into account the equalities (2.5) $\div$ (2.10) we have $J'^2(X) = -X$, $g(J'X, J'Y) = -g(X, Y)$, where $X, Y \in TM$. Hence, J' is an almost complex structure on M and the restriction of g on M is B-metric. Thus, the submanifold (M, J', g) $(\dim M = 2n)$ of $(\bar{M}, J, g)$ $(\dim \bar{M} = 2n + 2)$ is an almost complex manifold with B-metric.

Denoting by $\bar{\nabla}$ and ∇ the Levi-Civita connections of the metric g on $\bar{M}$ and M, respectively, the formulas of Gauss and Weingarten are

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma(X, Y) \quad X, Y \in TM;$$

$$\bar{\nabla}_X N_1 = -A_{N_1}X + D_X N_1; \quad \bar{\nabla}_X N_2 = -A_{N_2}X + D_X N_2; \quad X \in TM,$$

where σ is the second fundamental form on M, A_{N_i} $(i = 1, 2)$ is the second fundamental tensor with respect to N_i $(i = 1, 2)$ and D is the normal connection of M. Having in mind the properties of $\bar{\nabla}$ and (2.1), from the formulas of Gauss and Weingarten we compute

$$\sigma(X, Y) = -g(A_{N_1}X, Y)N_1 + g(A_{N_2}X, Y)N_2 = -g(X, A_{N_1}Y)N_1 + g(X, A_{N_2}Y)N_2;$$

$$D_X N_1 = \gamma(X)N_2; \quad D_X N_2 = \gamma(X)N_1,$$

where γ is an 1-form on M. Then the formulas of Gauss and Weingarten become

$$\bar{\nabla}_X Y = \nabla_X Y - g(A_{N_1}X, Y)N_1 + g(A_{N_2}X, Y)N_2; \tag{2.12}$$

$$\bar{\nabla}_X N_1 = -A_{N_1}X + \gamma(X)N_2; \quad \bar{\nabla}_X N_2 = -A_{N_2}X + \gamma(X)N_1.$$

3. Submanifolds of codimension 2 of a Kaehler manifold with B-metric. From now on, in this section the ambient almost complex manifold with B-metric $(\bar{M}, J, g)$ will be a Kaehler manifold with B-metric and (M, J', g) will be a submanifold of $\bar{M}$ of codimension 2 such that $J(T_pM)^{\perp} \subset T_pM$, $J(T_pM) \subset T_pM \oplus (T_pM)^{\perp}$ for each $p \in M$. Then $\bar{\nabla} J = 0$ and using (2.11), (2.12) we obtain

$$(\nabla_X J')Y = \eta^1(Y)\{A_{N_1}X - \nabla_X \xi_2\} + \eta^2(Y)\{A_{N_2}X + \nabla_X \xi_1\} + \{(\nabla_X \eta^2)Y - g(A_{N_1}X, Y)\}\xi_1 - \{(\nabla_X \eta^1)Y - g(A_{N_2}X, Y)\}\xi_2. \tag{3.1}$$

From $(\bar{\nabla}_X J)N_1 = (\bar{\nabla}_X J)N_2 = 0$, (2.4), (2.6), (2.12) we find

$$\begin{gathered} \nabla_X \xi_1 = -\varphi(A_{N_1}X) + \gamma(X)\xi_2, \ \ \nabla_X \xi_2 = -\varphi(A_{N_2}X) + \gamma(X)\xi_1, \\ g(A_{N_i}X, \xi_i) = 0 \iff A_{N_i}\xi_i = 0 \ \ (i = 1, 2), \\ g(A_{N_1}X, \xi_2) = -g(A_{N_2}X, \xi_1) \iff A_{N_1}\xi_2 = -A_{N_2}\xi_1. \end{gathered} \tag{3.2}$$

Using the equalities (2.6), (2.11), (3.1) and (3.2) we arrive to the following assertion

Theorem 3.1. *Let (M, J', g) be a submanifold of $(\bar{M}, J, g)$. Then*

$$\begin{aligned} F'(X, Y, Z) = {} & \eta^1(Y)\{g(A_{N_1}X, Z) + g(A_{N_2}X, J'Z)\} + \eta^2(Y)\{g(A_{N_2}X, Z) \\ & - g(A_{N_1}X, J'Z)\} + \eta^1(Z)\{g(A_{N_1}X, Y) + g(A_{N_2}X, J'Y)\} \\ & + \eta^2(Z)\{g(A_{N_2}X, Y) - g(A_{N_1}X, J'Y)\} + 2\gamma(X)\{\eta^1(Y)\eta^1(Z) + \eta^2(Y)\eta^2(Z)\} \end{aligned} \tag{3.3}$$

for arbitrary $X, Y, Z \in TM$.

Let $\bar{R}$ and R be the curvature tensors of $\bar{M}$ and M respectively. The curvature tensor of the normal connection D is denoted by $R^{\perp}$ and $R^{\perp}(X, Y, V) = D_X D_Y V - D_Y D_X V - D_{[X,Y]}V$, where $X, Y \in TM, V \in (TM)^{\perp}$. According to [3] the Gauss, Codazzi and Ricci equations are

$$\bar{R}(X, Y, Z, W) = R(X, Y, Z, W) + \pi_1(A_{N_1}X, A_{N_1}Y, Z, W) - \pi_1(A_{N_2}X, A_{N_2}Y, Z, W), \tag{3.4}$$

$$\begin{aligned} \bar{R}(X, Y, Z)^{\perp} = {} & \{g((\nabla_Y A)_{N_1}X, Z) - g((\nabla_X A)_{N_1}Y, Z)\}N_1 \\ & + \{g((\nabla_X A)_{N_2}Y, Z) - g((\nabla_Y A)_{N_2}X, Z)\}N_2, \end{aligned} \tag{3.5}$$

$$\bar{R}(X, Y, N_1, N_2) = d\gamma(X, Y) + g(A_{N_2}(A_{N_1}X), Y) - g(A_{N_1}(A_{N_2}X), Y), \tag{3.6}$$

where $\pi_1(X, Y, Z, W) = g(Y, Z)g(X, W) - g(X, Z)g(Y, W)$, $d\gamma(X, Y) = (\nabla_X \gamma)Y - (\nabla_Y \gamma)X$, $X, Y, Z, W \in TM$, $N_1, N_2 \in (TM)^{\perp}$.

Theorem 3.2. *The submanifold (M, J', g) of $(\bar{M}, J, g)$ belongs to the class $W_2 \oplus W_3$ iff*

$$\gamma(\xi_1) = \frac{1}{2}\{tr(A_{N_1}) + tr(J' \circ A_{N_2})\}; \ \ \gamma(\xi_2) = \frac{1}{2}\{tr(J' \circ A_{N_1}) - trA_{N_2}\}. \tag{3.7}$$

Proof. From (3.3) we find the Lee form θ for M $\theta(Z) = \{tr(A_{N_2}) - tr(J' \circ A_{N_1}) + 2\gamma(\xi_2)\}\eta^1(Z) - \{tr(A_{N_1}) + tr(J' \circ A_{N_2}) - 2\gamma(\xi_1)\}\eta^2(Z)$. It is known [2] that $M \in W_2 \oplus W_3$ iff $\theta = 0$ iff $-\{tr(A_{N_2}) - tr(J' \circ A_{N_1}) + 2\gamma(\xi_2)\}\xi_1 - \{tr(A_{N_1}) + tr(J' \circ A_{N_2}) - 2\gamma(\xi_1)\}\xi_2 = 0$. Since ξ_1, ξ_2 are linearly independent, the last equality is true iff (3.7) is valid.

Theorem 3.3. *Let A_{N_1} and A_{N_2} commute with J'. Then the submanifold (M, J', g) of $(\bar{M}, J, g)$ belongs to the class $W_1 \oplus W_2$ iff*

$$\gamma(X) = -\gamma(\xi_1)\eta^1(X) - \gamma(\xi_2)\eta^2(X). \tag{3.8}$$

Proof. According to [2] $M \in W_1 \oplus W_2$ iff $F'(X,Y,J'Z)+F'(Y,Z,J'X)+F'(Z,X,J'Y) = 0$ iff the following equality is valid

$$\gamma(X)\{\eta^1(Z)\eta^2(Y) - \eta^1(Y)\eta^2(Z)\} + \gamma(Y)\{\eta^1(X)\eta^2(Z) - \eta^1(Z)\eta^2(X)\} + \gamma(Z)\{\eta^1(Y)\eta^2(X) - \eta^1(X)\eta^2(Y)\} = 0. \tag{3.9}$$

If we substitute ξ_1 for Y and ξ_2 for Z in (3.9) we obtain (3.8). For the converse, let (3.8) be valid. Then by direct verification we obtain (3.9).

Taking into account the characteristic condition of the class W_2 from [2] and Theorem 3.2, Theorem 3.3 we get

Theorem 3.4. *Let A_{N_1} and A_{N_2} commute with J'. Then the submanifold (M, J', g) of $(\bar{M}, J, g)$ belongs to the class W_2 iff the equalities* (3.7) *and* (3.8) *are valid.*

Let us recall [3]: the normal connection D of M is said to be flat if $R^{\perp} = 0$.

Theorem 3.5. *The normal connection of the submanifold (M, J', g) of $(\bar{M}, J, g)$ is flat iff the* 1*-form γ is closed.*

Proof. The formulas of Gauss and Weingarten imply

$$R^{\perp}(X, Y, N_i) = \{(\nabla_X \gamma)Y - (\nabla_Y \gamma)X\}N_j, \quad i \neq j \;\; i, j = 1, 2. \tag{3.10}$$

If V is an arbitrary vector field normal to M, then $V = aN_1 + bN_2$, where $a, b \in F\bar{M}$ and

$$R^{\perp}(X, Y, V) = aR^{\perp}(X, Y, N_1) + bR^{\perp}(X, Y, N_2). \tag{3.11}$$

Since N_1, N_2 are linearly independent, from (3.10) and (3.11) we have $R^{\perp} = 0$ iff $(\nabla_X \gamma)Y - (\nabla_Y \gamma)X = d\gamma(X, Y) = 0$, i.e. γ is closed.

REFERENCES

[1] A. Borisov, G. Ganchev. Curvature properties of Kaehlerian manifolds with B-metric. *Math. and Education in Math.*, **14** (1985), 220–226.

[2] G. Ganchev, A. Borisov. Note on almost complex manifolds with Norden metric. *C. R. Acad. Bulg. Sci.*, **39** (1986), 31–34.

[3] K. Yano, M. Kon. CR-Submanifolds of Kaehlerian and Sasakian manifolds. *Progress in mathematics*, vol. **30**, Birkhauser, Boston, Basel, Stuttgart, 1983.

[4] К. Грибачев. Подмногообразия на В-многообразия. *ПУ "Паисий Хилендарски", Научни трудове – Математика*, том **22**, кн. 2 (1984), 249–263.

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НЯКОИ ПОДМНОГООБРАЗИЯ НА ПОЧТИ КОМПЛЕКСНИ МНОГООБРАЗИЯ С B-МЕТРИКА С КОРАЗМЕРНОСТ ДВЕ

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Подмногообразия на почти комплексни многообразия с B-метрика с коразмерност 2 в случая, когато нормалната площадка е напълно реална са разгледани. Тези подмногообразия не са холоморфни. Почти комплексна структура и B-метрика върху подмногообразията са дефинирани. Класът на някои подмногообразия на Келерово многообразие с B-метрика е намерен.