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ON THE RESTRICTED DOMINATION IN GRAPHS*

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An upper bound in terms of order and degrees on the restricted domination number of claw-free and net-free graphs is obtained.

1. Introduction. For the terminology in graph theory not presented here, we follow Haynes, et al. [4]. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. For any vertex v of G its *open neighborhood* $N(v,G)$ is $\{x \in V(G) | \ vx \in E(G)\}$, its *closed neighborhood* $N[v,G]$ is $N(v,G) \cup \{v\}$, and its *degree* $\deg(v,G)$ is $|N(v,G)|$. The minimum and maximum degrees of vertices in $V(G)$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a set $S \subseteq V(G)$ its *open neighborhood* $N(S,G)$ is $\cup_{v\in S} N(v,G)$, its *closed neighborhood* $N[S,G]$ is $N(S,G) \cup S$, and its *degree* $\deg(S,G)$ is $|N(S,G)\backslash S|$. The *k-set minimum degree* of G is the greatest integer $\delta_k(G)$ such that $\delta_k(G) \leq \deg(X,G)$ for all subsets X of $V(G)$ of cardinality k. The *k-set maximum degree* of G is the smallest integer $\Delta_k(G)$ such that $\Delta_k(G) \geq \deg(X,G)$ for all subsets X of $V(G)$ of cardinality k. The subgraph induced by $S \subseteq V(G)$ is denoted by $\langle S, G\rangle$. The *corona* of two graphs G_1 and G_2 is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2. The complete bipartite graph $K_{1,3}$ is called a *claw* and the graph $K_3 \circ K_1$ is called a *net.* A set D of vertices in G is a *dominating set* if $N[D,G] = V(G)$. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality taken over all dominating sets of G. A dominating set with $\gamma(G)$ vertices is called *γ-set.* The problem of determining $\gamma(G)$ for an arbitrary graph is NP-complete (Garey et al. [3]). Various authors have investigated bounds on the domination number of a graph in terms of order. The earliest such result is due to Ore [7]. McCuaig and Shepherd [6] investigated upper bounds on $\gamma(G)$ in the case $\delta(G) \geq 2$.

Theorem A. *Let G be a graph.*

(a) ([7]) *If $\delta(G) \geq 1$, then $\gamma(G) \leq |V(G)|/2$.*

(b) ([6]) *If G is a connected graph of order at least 8 and $\delta(G) \geq 2$ then $\gamma(G) \leq 2|V(G)|/5$.*

In this paper, we study restricted domination in graphs. The concept of restricted domination was introduced by Sanchis [8]. We shall use the notation which was proposed by Henning [5]. Let U be a subset of vertices of a graph G. The *restricted domination number $r(G,U,\gamma)$ of U* is the minimum cardinality of a dominating set of G containing

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U. A smallest possible dominating set of G containing all the vertices in U is called a γ_U*-set*. The *k-restricted domination number* of G is the smallest integer $r_k(G,\gamma)$ such that $r_k(G,U,\gamma) \le r_k(G,\gamma)$ for all subsets U of $V(G)$ of cardinality k. In the case $k=0$, the k-restricted domination number is the domination number.

Henning [5] extends the bounds on the domination number obtained in Theorem A to the restricted domination number:

Theorem B ([5])**.** *Let G be a connected graph and $1 \le k \le |V(G)|$.*
(a) If $\delta(G) \ge 1$, then $2r_k(G,\gamma) \le |V(G)| + k$;
(b) If $\delta(G) \ge 2$, then $5r_k(G,\gamma) \le 2|V(G)| + 3k$.

2. Bounds in terms of order and degrees. In this section, we extend the following bound on the domination number to the restricted domination number.

Theorem C (Cockayne, Ko and Shepherd [1])**.** *If a connected graph G is claw-free and net-free, then $\gamma(G) \le (|V(G)|+2)/3$.*

We shall need the following lemma.

Lemma 2.1. *Let G be a graph, $\delta(G) \ge 1$, $\emptyset \ne X \subseteq V_0 \subseteq V(G)$ and $Z_0 \ne \emptyset$ be the set of isolated vertices of $G_0 = G - V_0$. Let $D \subseteq N(Z_0,G)$ be minimal with respect to the property $Z_0 \subseteq N(D,G)$. Then:*

(a) [2] $2|D| \le |Z_0| + |N(Z_0,G)| \ / \ \delta(G)$;

(b) Let at most one of the components of G_0 have at least two vertices and let $G_1 = \langle N[Z_0,G],G\rangle$.

(b1) If G_0 is claw-free and net-free then $3r(G,X,\gamma) \le 3r(\langle V_0,G\rangle,X,\gamma) + 3|D| + |V(G)| - |V_0| - |Z_0| + 2$

(b2) If G_1 is claw-free graph then $|Z_0| \le 2|N(Z_0,G)|/\delta(G)$.

Proof. (b1) Let P be a γ_X-set of a graph $\langle V_0,G\rangle$ and Q be a γ-set of a graph $\langle V(G) - (V_0 \cup Z_0),G\rangle$. Then the set $S = P \cup Q \cup D$ is a dominating set of G and $X \subset S$. Hence $r(G,X,\gamma) \le |S| \le |P| + |Q| + |D|$ and from Theorem C it follows $r(G,X,\gamma) \le r(\langle V_0,G\rangle,X,\gamma) + (|V(G)| - |V_0| - |Z_0| + 2)/3 + |D|$. Hence we have the result.

(b2) Let $M \subseteq E(G)$ be the set of all edges between Z_0 and $N(Z_0,G)$. Then $|M| = \Sigma_{z\in Z_0} \deg(z,G) \ge \delta(G)|Z_0|$. On the other hand, since G_1 is claw-free, then $1 \le |N(a,G) \cap Z_0| \le 2$ for each $a \in N(Z_0,G)$. Hence $|M| \le 2|N(Z_0,G)|$. Therefore $\delta(G)|Z_0| \le 2|N(Z_0,G)|$ and the result follows. □

Theorem 2.2. *Let G be claw-free and net-free graph, $\delta(G) \ge 1$, $\emptyset \ne X \subseteq V(G)$, Z_0 be the set of isolated vertices of a graph $G_0 = G - N[X,G]$ and let G_0 have at most one component with at least two vertices.*

(i) If $Z_0 = \emptyset$ then $3r(G,X,\gamma) \le |V(G)| + 2|X| - \deg(X,G) + 2$.

(ii) If $Z_0 \ne \emptyset$ then $6r(G,X,\gamma) \le 2|V(G)| + 4|X| + 4 + \deg(X,G)/(5/\delta(G) - 2)$.

Proof. Let $V_0 = N[X,G]$. Then $r(\langle V_0,G\rangle,X,\gamma) = |X|$ and $|V_0| = \deg(X,G) + |X|$.

(i): If $V_0 = V(G)$ then the result is obvious. Now, let $V_0 \neq V(G)$ and let M be a γ-set of G_0. Then $X \cup M$ is a dominating set of G. Hence by Theorem C: $r(G, X, \gamma) \leq |X| + |M| \leq |X| + (|V(G)| - |X| - \deg(X, G) + 2)/3$ and the result follows.

(ii): From Lemma 2.1 (a) and ($b2$) we have: $6|D| - 2|Z_0| \leq 3|Z_0| + 3|N(Z_0, G)|/\delta(G) - 2|Z_0| \leq 5|N(Z_0, G)|/\delta(G)$. From this and from Lemma 2.1 ($b1$) it follows: $6r(G, X, \gamma) \leq 6|X| + 6|D| + 2|V(G)| - 2\deg(X, G) - 2|X| - 2|Z_0| + 4 = 4|X| + 2|V(G)| - 2\deg(X, G) + 4 + 6|D| - 2|Z_0| \leq 2|V(G)| + 4|X| + 4 - 2\deg(X, G) + 5|N(Z_0, G)|/\delta(G)$. Now, the result follows because of $N(Z_0, G) \subseteq N(X, G)\backslash G$. $\square$

Corollary 2.3. *Let G be a claw-free and net-free graph, $\delta(G) \geq 1$ and $1 \leq k \leq |V(G)|$. Let for each vertex set $X \subseteq V(G)$ of cardinality k, the graph $G_0 = G - N[X, G]$ have at most one component of order at least two.*

(i) *If* $\delta(G) = 1$ *then* $6r_k(G, \gamma) \leq 2|V(G)| + 4k + 4 + 3\Delta_k(G)$;

(ii) *If* $\delta(G) = 2$ *then* $6r_k(G, \gamma) \leq 2|V(G)| + 4k + 4 + \Delta_k(G)/2$;

(iii) *If* $\delta(G) \geq 3$ *then* $6r_k(G, \gamma) \leq 2|V(G)| + 4k + 4 - \delta_k(G)(2 - 5/\delta(G))$.

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ЗА ОГРАНИЧЕНОТО ДОМИНИРАНЕ В ГРАФИ

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Намерена е горна граница за числото на ограничено доминиране на $(K_{1,3}, K_3 \circ K_1)$-свободни графи.