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MEASURABILITY OF SETS OF PAIRS OF INTERSECTING NONISOTROPIC AND ISOTROPIC STRAIGHT LINES IN THE SIMPLY ISOTROPIC SPACE*

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In this paper we study the measurability of sets of pairs of intersecting nonisotropic and isotropic straight lines with respect to the group of the general similitudes and some of its subgroups. Some Crofton type formulas are also obtained.

1. Introduction. The simply isotropic space ${I_3}^{(1)}$ is defined as a projective space $\mathbb{P}_3(\mathbb{R})$ whith an absolute plane ω and two complex conjugate straight lines f_1, f_2 into ω with a (real) intersection point F [8], [10], [11]. All regular projectivities transforming the absolute figure into itself form the 8-parameter group G_8 of the general simply isotropic similitudes. Passing on to affine coordinates (x, y, z) each similitude of G_8 can be written in the form [8; p.3]

$$(1)\qquad \begin{aligned} x' &= c_1 + c_7(x\cos\varphi - y\sin\varphi),\\ y' &= c_2 + c_7(x\sin\varphi + y\cos\varphi),\\ z' &= c_3 + c_4x + c_5y + c_6z, \end{aligned}$$

where $c_1, c_2, c_3, c_4, c_5, c_6, c_7 > 0$ and φ are real parameters.

A straight line in $I_3^{(1)}$ is said to be (completely) isotropic if its infinite point coincides with the absolute point F; otherwise the straight line is said to be nonisotropic [8; p. 5].

We shall consider G_8 and the following its subgroups:

I. $B_7 \subset G_8 \iff c_7 = 1$. It is the group of the simply isotropic similitudes of the δ-distance [8; p. 5].

II. $S_7 \subset G_8 \iff c_6 = 1$. It is the group of the simply isotropic similitudes of the s-distance [8; p. 6].

III. $W_7 \subset G_8 \iff c_6 = c_7$. It is the group of the simply isotropic angular similitudes [8; p. 18].

IV. $G_7 \subset G_8 \iff \varphi = 0$. It is the group of the boundary simply isotropic similitudes [8; p. 8].

V. $V_7 \subset G_8 \iff c_6c_7^2 = 1$. It is the group of the volume preserving simply isotropic similitudes [8; p. 8].

VI. $G_6 = G_7 \cap V_7$. It is the group of the volume preserving boundary simply isotropic similitudes [8; p. 8].

VII. $B_6 = B_7 \cap G_7$. It is the group of the modular boundary motions [8; p. 9].

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VIII. $B_5 = B_7 \cap S_7 \cap G_7$. It is the group of the unimodular boundary motions [8; p. 9].

We emphasize that most of the common material of the geometry of the simply isotropic space ${I_3}^{(1)}$ can be found in [8], [10] and [11].

Using some basic concepts of the integral geometry in the sense of M. I. Stoka [9], G. I. Drinfel'd and A. V. Lucenko [5], [6], [7] we study the measurability of sets of pairs of intersecting nonisotropic and isotropic straight lines in ${I_3}^{(1)}$ with respect to G_8 and the subgroups indicated above. Analogous problems for sets of pairs of skew nonisotropic straight lines in ${I_3}^{(1)}$ have been treated in [2] and [3].

2. Measurability with respect to G_8. Let (G, J) be a pair of intersecting nonisotropic and isotropic straight lines determined by the equations

$$G: \ x = \alpha(z - \nu) + \lambda, \ y = \beta(z - \nu) + \mu, \qquad |\alpha| + |\beta| \neq 0 \tag{2}$$

and

$$J: \ x = \lambda, \quad y = \mu, \tag{3}$$

respectively. We can choose $\alpha \neq 0$ and then we can take the Plücker coordinates [7, p. 38–41] $p_2, p_3, p_5, p_6, q_4, q_5$ as the parameters of the set of pairs (G, J), where

$$p_2 = \frac{\beta}{\alpha}, \quad p_3 = \frac{1}{\alpha}, \quad p_5 = \frac{\alpha\nu - \lambda}{\alpha}, \quad p_6 = \frac{\beta\lambda - \alpha\mu}{\alpha}, \quad q_5 = -\lambda. \tag{4}$$

Under the action of (1) the pair $(G, J)(p_2, p_3, p_5, p_6, q_5)$ is transformed into the pair $(G', J')({p_2}', {p_3}', {p_5}', {p_6}', {q_5}')$ as follows:

$$\begin{aligned} {p_2}' &= c_7 K \ (\sin\varphi + p_2 \cos\varphi), \\ {p_3}' &= \quad K \ (c_4 + c_5 p_2 + c_6 p_3), \\ {p_5}' &= K\{(c_3 - c_5 p_6 + c_6 p_5) c_7 \cos\varphi - \\ &\qquad [\, c_3 + c_4 p_6 + c_6 (p_2 p_5 + p_3 p_6)] c_7 \sin\varphi - c_1 (c_4 + c_5 + c_6 p_3)\}, \\ {p_6}' &= c_7 K [(c_1 p_2 - c_2)\cos\varphi + (c_1 + c_2 p_2)\sin\varphi + c_7 p_6], \\ {q_5}' &= c_7 (q_4 \sin\varphi + q_5 \cos\varphi) - c_1, \end{aligned} \tag{5}$$

where $K = [c_7(\cos\varphi - p_2 \sin\varphi)]^{-1}$. The transformations (5) form the associated group $\overline{G}_8$ of G_8 [9, p. 34]. The group $\overline{G}_8$ is isomorphic to G_8 and the density with respect to G_8 of the pairs (G, J), if it exists, coincides with the density with respect to $\overline{G}_8$ of the points $(p_2, p_3, p_5, p_6, q_5)$ in the set of parameters.

The associated group $\overline{G}_8$ has the infinitesimal operators

$$\begin{aligned} &Y_1 = -p_3 \frac{\partial}{\partial p_5} + p_2 \frac{\partial}{\partial p_6} - \frac{\partial}{\partial q_5}, \quad Y_2 = -\frac{\partial}{\partial p_6}, \quad Y_3 = \frac{\partial}{\partial p_5}, \\ &Y_4 = -p_3 \frac{\partial}{\partial p_3} + p_6 \frac{\partial}{\partial p_6} + q_5 \frac{\partial}{\partial q_5}, \\ &Y_5 = (1 + p_2^2)\frac{\partial}{\partial p_2} + p_2 p_3 \frac{\partial}{\partial p_3} - p_3 p_6 \frac{\partial}{\partial p_5} + p_2 p_6 \frac{\partial}{\partial p_6} + (-p_6 - p_2 q_5)\frac{\partial}{\partial q_5}, \\ &Y_6 = \frac{\partial}{\partial p_3}, \quad Y_7 = p_2 \frac{\partial}{\partial p_3} - p_6 \frac{\partial}{\partial p_5}, \quad Y_8 = p_3 \frac{\partial}{\partial p_3} + p_5 \frac{\partial}{\partial p_5}. \end{aligned} \tag{6}$$

The group $\overline{G}_8$ acts transitively on the set of points $(p_2, p_3, p_5, p_6, q_5)$. The infinitesimal operators Y_1, Y_2, Y_3, Y_5 and Y_6 are arcwise unconnected but

$Y_4 = -q_5Y_1 - (p_2q_5 + p_6)Y_2 - p_3q_5Y_3 - p_3Y_6, \quad Y_7 = -p_6Y_3 + p_2Y_6, \quad Y_8 = p_5Y_3 + p_3Y_6.$

Since $Y_3(p_5) + Y_6(p_3) \neq 0$ we establish the following

Theorem 2.1. *A set of pairs of intersecting nonisotropic and isotropic straight lines is not measurable with respect to the group G_8 and it has no measurable subsets.*

3. Measurability with respect to B_5. The associated group $\overline{B}_5$ of the group B_5 has the infinitesimal operators Y_1, Y_2, Y_3, Y_6 and Y_7 given by (6). The group $\overline{B}_5$ acts intransitively on the set of parameters $(p_2, p_3, p_5, p_6, q_5)$ and therefore the set of pairs (G, J) is not measurable with respect to B_5. The system

$$Y_1(f) = 0, \quad Y_2(f) = 0, \quad Y_3(f) = 0, \quad Y_6(f) = 0, \quad Y_7(f) = 0$$

has the solution $f = p_2$ and it is an absolute invariant of $\overline{B}_5$. Consider the subset of pairs (G, J) satisfying the condition

$$p_2 = h, \tag{7}$$

where $h = \text{const}$. The group $\overline{B}_5$ induces the group on the subset (7) $B_5^\star$ with the infinitesimal operators

$$Z_1 = -p_3\frac{\partial}{\partial p_5} + h\frac{\partial}{\partial p_6} - \frac{\partial}{\partial q_5}, \quad Z_2 = -\frac{\partial}{\partial p_6}, \quad Z_3 = \frac{\partial}{\partial p_5},$$

$$Z_6 = \frac{\partial}{\partial p_3}, \quad Z_7 = h\frac{\partial}{\partial p_3} - p_6\frac{\partial}{\partial p_5}$$

and obviously it is transitive. The Deltheil system [4; p. 28]

$$Z_1(f) = 0, \; Z_2(f) = 0, \; Z_3(f) = 0, \; Z_6(f) = 0, \; Z_7(f) = 0$$

has the solution $f = c$, where $c = \text{const}$.

Thus we conclude that it holds

Theorem 3.1. *The set of pairs $(G, J)(p_2, p_3, p_5, p_6, q_5)$ is not measurable with respect to the group B_5 but it has the measurable subset (7) with the density*

$$d(G, J) = dp_3 \wedge dp_5 \wedge dp_6 \wedge dq_5. \tag{8}$$

Differentiating (4) and substituting into (8) we obtain another expression for the density:

Corollary 3.1. *The set of pairs $(G, J)(\alpha, \beta, \lambda, \mu, \nu)$, determined by (2) and (3), is not measurable with respect to the group B_5 but it has the measurable subset*

$$\frac{\beta}{\alpha} = h, \qquad h = \text{const}$$

with the density

$$d(G, J) = \frac{1}{\alpha^2}\, d\alpha \wedge d\lambda \wedge d\mu \wedge d\nu.$$

4. Some Crofton type formulas with respect to B_5. Since $f = \dfrac{\beta}{\alpha}$ is an absolute invariant of the pairs (G, J) with respect to B_5, it follows that we can define the density for the set of pairs $(G, J)(\alpha, \beta, \lambda, \mu, \nu)$ of intersecting nonisotropic and isotropic straight lines by the equality

$$d(G, J) = \left(\frac{\beta}{\alpha}\right)^2 d\alpha \wedge d\beta \wedge d\lambda \wedge d\mu \wedge d\nu. \tag{9}$$

Let G_i and J_i, $i = 1, 2$, be the projections of G and J into Oxz and Oyz obtained in a parallel way to Oy and Ox, respectively. From here it follows

$$G_1 : \quad z = \frac{1}{\alpha}x + \nu - \frac{\lambda}{\alpha}, \; y = 0,$$

and

$$G_2 : \quad z = \frac{1}{\beta}y + \nu - \frac{\mu}{\beta}, \; x = 0$$

Then

$$d(G_1, J_1) = dG_1 \wedge d\lambda,$$

where $d(G_1, J_1)$ is the density for the pairs (G_1, J_1) in the isotropic plane Oxz under the group ${}^1H_3^3$ [1; p. 198] and dG_1 is the density of the straight lines in Oxz under the metric group. We note that the group ${}^1H_3^3$ is the restriction of the group B_5 on Oxz. Analogously,

$$d(G_2, J_2) = dG_2 \wedge d\mu$$

is the density for the pairs (G_2, J_2) in the isotropic plane Oyz under the group ${}^2H_3^3$, which is the restriction of the group B_5 on the plane Oyz and dG_2 is the density of the straight lines in Oyz under the metric group. We have

$$dG_1 = d(\frac{1}{\alpha}) \wedge d(\nu - \frac{\lambda}{\alpha}) \tag{10}$$

and

$$dG_2 = d(\frac{1}{\beta}) \wedge d(\nu - \frac{\mu}{\beta}). \tag{11}$$

By exterior multiplication of (10), (11) with $d\lambda$, $d\mu$, respectively, we get

$$d(G_1, J_1) = \frac{1}{\alpha^2}\, d\alpha \wedge d\lambda \wedge d\nu \tag{12}$$

and

$$d(G_2, J_2) = \frac{1}{\beta^2}\, d\beta \wedge d\mu \wedge d\nu. \tag{13}$$

Replacing (10), (11), (12) and (13) into (9) we obtain

$$d(G, J) = |\beta^5|\, d(G_1, J_1) \wedge dG_2 = \beta^4 |\alpha|\, dG_1 \wedge d(G_2, J_2). \tag{14}$$

Thus we established the following

Theorem 4.1. *The density for the set of pairs (G, J) of the intersecting nonisotropic and isotropic straight lines with respect to the group B_5 satisfies the relations* (14)*.*

5. Measurability with respect to $\mathbf{B_7, S_7, W_7, G_7, V_7, G_6}$ and $\mathbf{B_6}$. By arguments similar to the ones used above we examine the measurability of sets of pairs (G, J) with respect to all the rest groups. We summarize the results in the following

Theorem 5.1. *A set of pairs of intersecting nonisotropic and isotropic straight lines is not measurable and it has no measurable subsets with respect to the groups B_7, S_7, W_7, G_7, V_7, G_6 and B_6.*

REFERENCES

[1] A. Borisov. Integral geometry in the Galilean plane. Research work qualifying for a degree full-professor, Sofia, 1998 (in Bulgarian).

[2] A. Borisov, M. Spirova. Measurability of sets of pairs of skew nonisotropic straight lines of type alpha in the simply isotropic space. (to appear)

[3] A. Borisov, M. Spirova. Measurability of sets of pairs of skew nonisotropic straight lines of type beta in the simply isotropic space. (to appear)

[4] R. Deltheil. Sur la théorie des probabilité géométriques. *Thése Ann. Fac. Sc. Univ. Toulouse*, (3), **11** (1919), 1–65.

[5] G. I. Drinfel'd. On the measure of the Lie groups. *Zap. Mat. Otdel. Fiz. Mat. Fak. Kharkov. Mat. Obsc.*, **21** (1949), 47–57 (in Russian).

[6] G. I. Drinfel'd, A. V. Lucenko. On the measure of sets of geometric elements. *Vest. Kharkov. Univ.*, **31** (1964), No 3, 34–41 (in Russian).

[7] A. V. Lucenko. On the measure of sets of geometric elements and their subsets. *Ukrain. Geom. Sb.*, **1** (1965), 39–57 (in Russian).

[8] H. Sachs. Isotrope Geometrie des Raumes. Friedr. Vieweg and Sohn, Braunschweig/Wiesbaden, 1990.

[9] M. I. Stoka. Geometrie Integrala. Ed. Acad. RPR, Bucuresti, 1967.

[10] K. Strubecker. Differentialgeometrie des isotropen Raumes I. *Sitzungsber. Ősterr. Akad. Wiss. Wien*, **150** (1941), 1–53.

[11] K. Strubecker. Differentialgeometrie des isotropen Raumes II, III, IV, V. *Math. Z.*, **47** (1942), 743–777; **48** (1942), 369–427; **50** (1944), 1–92; **52** (1949), 525–573.

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ИЗМЕРИМОСТ НА МНОЖЕСТВА ОТ ДВОЙКИ КРЪСТОСАНИ НЕИЗОТРОПНА И ИЗОТРОПНА ПРАВА В ПРОСТО ИЗОТРОПНО ПРОСТРАНСТВО

Адриян В. Борисов, Маргарита Г. Спирова

В статията е изследвана измеримостта на множества от двойки пресичащи се неизотропна и изотропна права в просто изотропно пространство и са получени съответните гъстоти относно групата на подобностите и някои нейни подгрупи.