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ON THE COMMUTING OF CURVATURE OPERATORS*

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In the present note we characterize four-dimensional Riemannian manifolds for which some curvature operators commute at each point of the manifolds. This problem was stated by the first author, and it is a new idea for characterization of Riemannian and another classes of smooth manifolds. In the present note we obtain a result, which is one of the first in this field, namely, we characterize the four-dimensional Einstein manifolds under the hypothesis of commuting of some curvature operators.

Let (M, g) be an $n-$dimensional Riemannian manifold with a metric tensor g, p be a point of M, and M_p be the tangent space to Mat this point. In the Riemannian geometry a very important role plays the curvature tensor R of the manifold (M, g), which has the following properties

$$R(x,y,z,u) = -R(y,x,z,u), \tag{1}$$

$$R(x,y,z,u) = -R(x,y,u,z), \tag{2}$$

$$\sigma_{x,y,z}R(x,y,z,u) = 0 \text{ (the first Bianchi identity)}, \tag{3}$$

where in (3) σ denotes a cyclic sum. In the properties (1)–(3) of R x, y, z, u are arbitrary tangent vectors which belong to the tangent space M_p at a point $p \in M$.

Using the Riemannian curvature tensor R of a Riemannian manifold (M, g) it is possible to define some *curvature operators* in the tangent space M_p, at any point $p \in M$. We will use the following linear curvature operators:

1. Jacobi operator [1]

$$R_X(u) = R(X)(u) = R(u, X, X),$$

defined for each unit tangent vector $X \in M_p$ at a point $p \in M$;

2. Stanilov skew-symmetric curvature operator

$$S_\alpha(u) = S_{X,Y}(u) = R(X, Y, u),$$

defined for each orthonormal basis X, Y of a two-dimensional tangent subspace α in M_p at a point $p \in M$;

3. Generalized Jacobi operator

$$R(E^m)(u) = \Sigma_{i=1,m}R(X_i)(u),$$

defined for any $m-$dimensional subspace $E^m \subset M_p$ at a point $p \in M$, where $\{X_i\}_{i=1,m}$ $(m < n)$ is an orthonormal basis in E^m.

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The curvature operators in 2. and 3. are defined in the Riemannian case by the first author. It is easy to prove that these curvature operators don't depend on the orthonormal bases in the corresponding tangent subspace. A new series of investigations were done in [2] under the assumption

$$S_\alpha \circ R_\alpha = R_\alpha \circ S_\alpha, \tag{4}$$

which must be true for any two-dimensional tangent subspace $\alpha \in M_p$ at any point $p \in M$. This condition is equivalent to the requirement$S_\alpha \circ R_\alpha$ to be a skew-symmetric curvature operator.

Further, we consider this problem again in the four-dimensional case. First example of a four-dimensional Riemannian manifold, which satisfies condition (4), is every space of constant sectional curvature K, with curvature tensor R of the form

$$R(x, y, z) = K(g(y, z)x - g(x, z)y), \tag{5}$$

for any tangent vectors x, y, z in M_p, at any point $p \in M$. Second example of four-dimensional Riemannian manifold which satisfies this condition is every four-dimensional Einstein manifold. These are manifolds such that

$$\rho(u, v) = c.g(u, v), \tag{6}$$

for some constant c on the whole manifold (M, g), where $\rho(u, v)$ is the Ricci tensor which can be defined by the equality $\rho(u, v) = \Sigma_{i=1,m} R(e_i, u, v, e_i)$, for any orthonormal basis $\{e_i\}_{i=1,n}$ in the tangent subspace M_p. It is clear that the manifolds which satisfy (5) satisfy (6) too.

In the present note we'll prove that in the four-dimensional case Einstein manifolds are the unique possible four-dimensional Riemannian manifolds which satisfy the hypothesis (4).

Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis in the tangent space M_p, at any point $p \in M$, and let in the general case $A = (a_{ij})$, $(i, j = 1, n)$ be the matrix of the skew-symmetric curvature operator $S_\alpha \circ R_\alpha$ for a two-dimensional tangent subspace α in M_p. If $\alpha = \operatorname{span}\{e_1, e_2\}$, then

$$(7)(A)=\begin{pmatrix} 0 & -2K_{12} & -R_{2113} & -R_{2114} \\ K_{12} & 0 & R_{1223} & R_{1224} \\ R_{2113} & -R_{1223} & 0 & R_{1234} \\ R_{2114} & -R_{1224} & -R_{1234} & 0 \end{pmatrix} \begin{pmatrix} K_{12} & 0 & R_{1223} & R_{1224} \\ 0 & K_{12} & R_{2113} & R_{2114} \\ R_{1223} & R_{2113} & K_{13}+K_{23} & R_{3114}+R_{3224} \\ R_{1224} & R_{2114} & R_{3114}+R_{3224} & K_{14}+K_{24} \end{pmatrix}$$

and hence for the entries of (A) we have the formulas

$$\begin{aligned} a_{11} &= -R_{2113}R_{1223} - R_{2114}R_{1224} \\ a_{22} &= R_{1223}R_{2113} + R_{1224}R_{2114} \\ a_{33} &= R_{1234}(R_{3114} + R_{3224}) \\ a_{44} &= -R_{1234}(R_{3114} + R_{3224}) \\ a_{12} &= -K_{12}^2 - R_{2113}^2 - R_{2114}^2 \\ a_{21} &= K_{12}^2 + R_{1223}^2 + R_{1224}^2 \\ a_{13} &= -K_{12}R_{2113} - (K_{13} + K_{23})R_{2113} - R_{2114}(R_{3114} + R_{2114}) \\ a_{31} &= K_{12}R_{2113} + R_{1234}R_{1224} \end{aligned} \tag{8}$$

$$
\begin{aligned}
a_{14} &= -K_{12}R_{2114} - R_{2113}(R_{3114} + R_{3224}) - R_{2114}(K_{14} + K_{24}) \\
a_{41} &= K_{12}R_{2114} - R_{1234}R_{1223} \\
a_{23} &= (K_{12} + K_{13} + K_{23})R_{1223} + R_{1224}(R_{3114} + R_{3224}) \\
a_{32} &= -K_{12}R_{1223} + R_{2114}R_{1234} \\
a_{24} &= K_{12}R_{1224} + R_{1223}(R_{3114} + R_{3224}) + (K_{14} + K_{24})R_{1224} \\
a_{42} &= -K_{12}R_{1224} - R_{1234}R_{2113} \\
a_{34} &= R_{2113}R_{1224} - R_{1223}R_{2114} + (K_{14} + K_{24})R_{1234} \\
a_{43} &= R_{2114}R_{1223} - R_{1224}R_{2113} - R_{1234}(K_{13} + K_{23}).
\end{aligned}
$$

Since A is a skew-symmetric matrix, then we have

(9) $$a_{ii} = 0, \quad (i = 1, 2, 3, 4), \quad \text{and} \quad a_{ij} = -a_{ji},$$

for any different indices $i, j = 1, 2, 3, 4$. Hence $a_{12} = -a_{21}$, which according to the corresponding above entries of (A), gives

(10) $$R^2_{2113} + R^2_{2114} = R^2_{1223} + R^2_{1224}.$$

Let e_1, e_2, e_3, e_4 be an orthonormal basis of eigenvectors of the Jacobi operator R_{e1} in the tangent space M_p, at a point $p \in M$ (that is possible since R_{e1} is a symmetric linear operator). In this case we have the relations

(11) $$R_{2113} = R_{2114} = R_{3114} = 0,$$

and now we have

$$R_{2113} = R_{2114} = R_{1223} = R_{1224} = 0.$$

Since the relation (4) is true for any 2-plane $E^2 = e_1 \wedge e_j$ $(j = 2, 3, 4)$, from the last equality we get (if 1 is fixed and we change the indices):

$(2 \leftrightarrow 3)$ $$R_{3112} = R_{3114} = R_{1332} = R_{1334} = 0$$

$(3 \leftrightarrow 4)$ $$R_{4112} = R_{4113} = R_{1442} = R_{1443} = 0.$$

Now from (11) it follows that

$$\rho_{12} = \rho_{13} = \rho_{14} = 0,$$

because of

$$
\begin{aligned}
\rho_{12} &= R_{1332} + R_{1442}, \\
\rho_{13} &= R_{1223} + R_{1443}, \\
\rho_{14} &= R_{1224} + R_{1334}.
\end{aligned}
$$

From the last three equalities, and using that ρ is a bilinear function on M_p, we can obtain the equation $\rho_{1x} = \rho\ (e_1, x) = 0$, which holds for any unit tangent vector $x \perp e_1$, $x \in M_p$, and a point $p \in M$.

Since e_1 is an arbitrary tangent vector in M, this relation is valid for any orthonormal pair of tangent vector x, e_1 in the tangent space M_p, at any point $p \in M$, and this means that (M, g) is an Einstein manifold.

Thus we proved our

Theorem 1. *Let (M, g) be a 4-dimensional Riemannian manifold. Then the following assertions are equivalent:*

a) *For any* 2*-dimentional tangent plane* $E^2 \in M_p$*, and for any point* p *of the manifold* (M, g) *relation* $k_{E^2} \circ S_{E^2} = S_{E^2} \circ k_{E^2} 2$ *holds;*

b) (M, g) *is an Einstein Riemannian manifold.*

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ВЪРХУ КОМУТИРАНЕТО НА КРИВИННИ ОПЕРАТОРИ

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Разглеждат се следните два кривинни оператора: кососиметричния оператор и обобщения оператор на Якоби за 2-мерно допирателно пространство в точка на 4-мерно Риманово многообразие. Поставя се проблема за характеризиране на такива многобразие при условие за тяхното комутиране. Доказва се, че те комутират точно когато многообразието е Айнщайново.