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FOUR-DIMENSIONAL RIEMANNIAN MANIFOLDS WITH COMMUTING STANILOV CURVATURE OPERATORS*

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We characterize locally the four-dimensional Rimannian manifolds of constant sectional curvature by applying a condition for commuting of the Stanilov skew-symmetric curvature operator and the generalized Jacobi operator of second order.

Let (M, g) be an n-dimensional Riemannian manifold with curvature tensor R of type (1, 3), or of type (0, 4), both related by the identity

$$g(R(x, y, z), u) = R(x, y, z, u). \tag{1}$$

Further we denote by M_p the tangent space to M at a point $p \in M$.

The classical *Jacobi operator*

$$J_x : M_p \to M_p, \tag{2}$$

induced by a unit tangent vector $x \in M_p$ is given by

$$J_x(u) = R(u, x, x). \tag{3}$$

It is a symmetric curvature operator.

In 1990 G. Stanilov [1, 2] defined the now well-known and widely used *skew-symmetric curvature operator*:

$$k_{E_2} : M_p \to M_p, \tag{4}$$

induced by a two-dimensional tangent subspace $E_2 \in M_p$, in the following way:

$$k_{E_2}(u) = R(x, y, u), \tag{5}$$

were (x, y) is an orthonormal basis of E_2.

It is easy to see that the curvature operator k_{E_2} is invariant with respect to any orthogonal transformation of E_2 with positive determinant.

Let E_m be an m-dimensional tangent subspace of the tangent space M_p.

In 1991 G. Stanilov [2, 3] defined the so-called *generalized Jacobi operator*:

$$S_{E_m} : M_p \to M_p, \tag{6}$$

by the formula

$$S_{E_m} = \sum_{i=1}^{m} J_{e_i}, \tag{7}$$

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where $\{e_i\}$, $i = 1, \dots, m$ is an orthonormal basis of E_m.

It is clear that S_{E_m} is invariant with respect to any orthogonal transfornation of E_m and it is a symmetric curvature operator.

Hence every m-dimensional tangent subspace $E_m \in M_p$ induces a corresponding generalized Jacobi operator S_{E_m} [4, 5, 6, 7].

Using these curvature operators, Stanilov and Tsankov [8] gave a characterization for their commuting in the tangent space M_p, at any point $p \in M$.

Based on these investigations, in the present paper we consider the class of four-dimensional Rimannian manifolds such that at any point $p \in M$ and for any tangent plane $E_2 \in M_p$ the following relation holds:

$$K_{\perp E_2} \circ S_{E_2} = S_{E_2} \circ k_{\perp E_2}, \tag{8}$$

where $\perp$ denotes the orthogonal complement of E_2 in M_p.

Remark 1. This condition is equivalent to the requirement that $k_{\perp E_2} \circ S_{E_2}$ is a skew-symmetric curvature operator [8].

Let (e_1, e_2, e_3, e_4) be an arbitrary orthonormal basis of the tangent space M_p at a point $p \in M$. Then the matrices of the curvature operators k_{e_1,e_2} and S_{e_3,e_4} with respect to this basis are the following:

$$(k_{e_1,e_2}) = \begin{pmatrix} 0 & -K_{12} & -R_{2113} & -R_{2114} \\ K_{12} & 0 & R_{1223} & R_{1224} \\ R_{2113} & -R_{1223} & 0 & R_{1234} \\ R_{2114} & -R_{1224} & -R_{1234} & 0 \end{pmatrix},$$

$$(S_{e_3,e_4}) = \begin{pmatrix} K_{13}+K_{14} & R_{1332}+R_{1442} & R_{1443} & R_{1334} \\ R_{1332}+R_{1442} & K_{23}+K_{24} & R_{2443} & R_{2334} \\ R_{1443} & R_{2443} & K_{34} & 0 \\ R_{1334} & R_{2334} & 0 & K_{34} \end{pmatrix}$$

Following Remark 1 we have that the matrix (a_{ij}) of the curvature operator $k_{e_1,e_2} \circ S_{e_3,e_4}$ is skew-symmetric. Thus we get:

$$\begin{aligned} a_{11} &= -K_{12}\rho_{12} - R_{2113}R_{1443} - R_{2114}R_{1334} = 0, \\ a_{22} &= K_{12}\rho_{12} + R_{1223}R_{2443} + R_{1224}R_{2334} = 0, \\ a_{33} &= R_{2113}R_{1443} - R_{1223}R_{2443} = 0, \\ a_{44} &= -R_{2114}R_{1334} + R_{1224}R_{2334} = 0. \end{aligned} \tag{9}$$

Here $K_{ij} = R(e_i, e_j, e_j, e_i)$ is the sectional curvature of the plane spanned by the tangent vectors e_i, e_j and $\rho_{uv} = \rho(u, v)$ is the Ricci tensor, for which

$$\rho(u, v) = \sum_{i=1}^{4} R(e_i, u, v, e_i).$$

In (9) we change the indeces 1 and 2. Thus we get:

$$
(10)\qquad
\begin{aligned}
&K_{12}\rho_{12}+R_{1223}R_{2443}+R_{1224}R_{2334}=0,\\
&K_{12}\rho_{12}+R_{2113}R_{1443}+R_{2114}R_{1334}=0,\\
&R_{1223}R_{2443}-R_{2113}R_{1443}=0,\\
&-R_{1224}R_{2334}+R_{2114}R_{1334}=0.
\end{aligned}
$$

Let x be a smooth tangent vector field on the manifold (M,g). Having in mind that any Jacobi operator J_x is a symmetric linear operator and following some results for this class of operators [9], we define the three submanifolds M_3, M_2, M_1 of M, consisting of the points $x\in M$ at which J_x has 3, 2 or 1 different, non-zero eigenvalues, respectively.

Let $x=e_1$ and e_1, e_2, e_3, e_4 be an orthonormal basis of eigenvectors of the Jacobi operator J_{e_1} in the tangent space M_p at a point $p\in M$. Then

$$
(11)\qquad R_{2113}=R_{2114}=R_{3114}=0
$$

and for the corresponding eigenvalues c_0, c_1, c_2, c_3 of the eigenvectors e_1, e_2, e_3, e_4 we have

$$
c_0=0,\quad c_1=K_{12},\quad c_2=K_{13},\quad c_3=K_{14}.
$$

From (11) and (9), we obtain the equality

$$
K_{12}\rho_{12}=0.
$$

Since the relation (8) is true for any 2-plane $E_2=e_1\wedge e_j$, we get the system

$$
(12)\qquad K_{1j}.\rho_{1j}=0,\qquad\qquad (j=2,3,4).
$$

Now we consider the case when the non-zero eigenvalues c_1, c_2, c_3 of the Jacobi operator J_{e_1} are different at a point $p\in M_3$. Then (12) implies

$$
(13)\qquad \rho_{1j}=0\qquad\qquad (j=2,3,4)
$$

from where it is easy to find that

$$
(14)\qquad \rho_{1x}=0,
$$

for any tangent vector $x\perp e_1$, at any point $p\in M_3$.

The second possibility is when the Jacoby operator J_{e_1} has two different non-zero eigenvalues at a point $p\in M_2$, say $c_2=K_{13}$ and $c_3=K_{14}$. Then from the system (12) we get

$$
(15)\qquad \rho_{13}=0,\qquad \rho_{14}=0.
$$

From (9), (10), (11) changing the index 1 by indeces 3 and 4, we get:

$$
(16)\qquad
\begin{aligned}
&K_{12}\rho_{12}=0,\\
&K_{23}\rho_{23}=0,\\
&K_{24}\rho_{24}=0.
\end{aligned}
$$

Suppose $\rho_{12}=0$. Then using (15) we obtain (13) and (14) again, but for any tangent vector $x\perp e_i$ at a point $p\in M_2$.

If $\rho_{12}=\rho_{11}-\rho_{22}\neq 0$, then

$$
(17)\qquad \rho_{11}\neq\rho_{22}.
$$

Since from (15) it follows that

(18) $$\rho_{11} - \rho_{33} = 0, \qquad \rho_{11} - \rho_{44} = 0,$$

we get from (17) and (18) that

(19) $$\rho_{23} \neq 0, \qquad \rho_{24} \neq 0.$$

These unequalities and (16) imply that

$$K_{23} = K_{24} = 0,$$

and then using $c_1 = K_{12} = 0$, we obtain

(20) $$\begin{aligned} \rho_{33} &= K_{13} + K_{23} + K_{24} = c_2 + K_{34}, \\ \rho_{44} &= K_{14} + K_{24} + K_{34} = c_3 + K_{34}. \end{aligned}$$

From the second equality in (15) we obtain

(21) $$\rho_{33} = \rho_{44},$$

and then from (20) and the equality (21) we get $c_2 = c_3$, a contradiction.

The third possibility is when the Jacobi operator J_{e_1} has just one non-zero eigenvalue c on M_1. If e_1, u_2, u_3, u_4 is an arbitrary orthonormal basis of the tangent space M_p at a point $p \in M_1$, then using the special form of the characteristic equation of the Jacobi operator J_{e_1} with respect to this basis we find

$$R_{2113} = 0,$$

which holds for any orthonormal triple e_1, u_2, u_3 in the tangent space M_p at a point $p \in M_1$. This equality means that the submanifold M_1 has a constant sectional curvature, and hence (14) is valid again, now for any tangent vector $x \perp e_1$, at a point $p \in M_1$.

Thus, the equality (14) is always true. More exactly the equality (14) holds good for any tangent vector $x \perp e_1$, at any point $p \in M_i$, $1 \leq i \leq k$, which means that all these submanifolds of the basic manifold M are Einstein submanifolds. Then the Riemannian manifold (M, g) is also an Einstein manifold. Now we apply Lemma 1 and Lemma 2 from the paper [8]. It follows that the operators $k_{E_2} = k$ and $S_{\perp E_2} = S$ are orthogonal and for any orthonormal basis (e_i), $i = 1, 2, 3, 4$ we have

$$g(k(e_i), S(e_j)) + g(k(e_j), S(e_i)) = 0,$$

for all $i, j = 1, 2, \ldots, n$. If (e_i) is the well-khown Singer-Thorpe basis then $R_{ijkl} = 0$, when three of the indeces are distinct. Hence $\rho = K_{ij}$ and the manifold is of constant sectional curvature.

Conversely, if the Riemannian manifold (M, g) is of constant sectional curvature, then the relation (8) holds true. Hence the following conditions are equivalent:

(1) The relation (8) holds true for any tangent plane $E_2 \in M_p$, $p \in M$;

(2) (M, g) is locally isometric to a Riemannian manifold of constant sectional curvature.

REFERENCES

[1] R. Ivanova, G. Stanilov. A skew-symmertic curvature operator in Riemannian geometry. Symposia Gaussiana, Conf. A (Eds Behara, Fritsch), Lintz, Berlin, New York, 1995, 391–395.

[2] G. Stanilov. Curvature operators based on the skew-symmetric curvature operator and their place in Differential geomettry. Preprint, 2000, 1–6.

[3] I. Petrova, G. Stanilov. A generalized Jacobi operator on the 4 dimensional Riemannian geometry. *Annuare de l'Universite de Sofia St. Kliment Ohridski*, livre 1 Mathematiques, **85** (1991), 50–56.
[4] S. Ivanov, I. Petrova. Riemannian manifold in which the skew-symmetric curvature operator has constant eigenvalue. *Geometriae Dedicata*, **70** (1998), 269–282.
[5] G. Stanilov, V. Videv. Four dimensional Osserman manifolds. *Abh. Math. Sem. Univ. Hamburg*, **68** (1998), 1–6.
[6] P. Gilkey. Generalized Osserman manifolds. *Abh. Math. Sem. Univ. Hamburg*, **68** (1998), 125–127.
[7] P. Gilkey, G. Stanilov, V. Videv. Pseudo-Riemannian manifolds whose generalized Jacobi operator has a constant characteristic polynomial. *Journal of Geometry (Basel)*, **62** (1998), 144-153.
[8] G. Stanilov, J. Tzankov. A common characterization od the Einstein and conformal flat manifolds. *Jounal of Geometry (Basel)*, (2003), to appear.
[9] T. Kato. Perturbation theory for linear operators. Berlin. Springer-Verlag, 1996.
[10] I. Singer, J. Thorpe. The curvature of 4-dimentional Einstein spaces. Global Analisys. Papers in honor of K. Kodaire. University of Tokyo press, Princenton University press, 1969.

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ЧЕТИРИМЕРНИ РИМАНОВИ МНОГООБРАЗИЯ С КОМУТИРАЩИ КРИВИННИ ОПЕРАТОРИ НА СТАНИЛОВ

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Получена е локална характеризация на Римановите многообразия с постоянна секционни кривина, като се използва едно условие за комутиране на антисиметричния кривинен оператор на Станилов и обобщения оператор на Якоби от втори ред.