МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2004
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2004
Proceedings of the Thirty Third Spring Conference of
the Union of Bulgarian Mathematicians
Borovets, April 1–4, 2004

APPROXIMATION BY OPERATORS OF CAO-GONSKA TYPE $G_{S,N}$ AND $G^*_{S,N}$. DIRECT AND CONVERSE THEOREMS

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The aim of this paper is to establish equivalence between the approximating rate of a linear process and the appropriate Peetre K-functional. A general method is applied (developed by Z. Ditzian and K. Ivanov) for proving converse inequalities.

Introduction. Let $f \in \mathbf{C}[-1,1]$, $n \in \mathbf{N}$, $s \in \mathbf{N}$,

$$G_{s,n}(f,x) = \pi^{-1} \int_{-\pi}^{\pi} f(\cos(\arccos x + v)) K_{s,n}(v) dv,$$

$$K_{s,n} = c_{n,s} \left(\frac{\sin(nv/2)}{\sin(v/2)}^{2s} \right), \qquad \pi^{-1} \int_{-\pi}^{\pi} K_{s,n}(v) dv = 1.$$

$L(f,x) = \dfrac{1}{2} f(1)(x+1) + \dfrac{1}{2} f(-1)(1-x)$, $-1 \le x \le 1$.

We consider the sequence of operators $G^*_{s,n} = G_{s,n} + L - G_{s,n} \circ L$.

For a normed linear space S and a dense subspace Y of X induced by the operator D and given by $Y = \{f \in X; Df \in X\}$ the Peetre K-functional is defined for $f \in X$ by $K(f,t) = \inf\limits_{g \in Y} (\|f-g\|_X + t\|Df\|_X)$, $t > 0$.

Generally, D can be thought as a differential operator, for instance $Df = f'$, $Df = \tilde{f}'$, $Df = \varphi f'$. In many situation in which we have a sequence of a family of uniformly bounded operators, i.e., $Q_n : X \to X$ with $\|Q_n\| \le M$, it is proved the equivalence $K(f, \lambda(n)) \sim \|f - Q_n f\|$.

We consider the operators $H_1(g(x)) \overset{\text{def}}{=} \left(1-x^2\right)^{\frac{1}{2}} \dfrac{d}{dx}(g(x)$, $H \overset{\text{def}}{=} (\ H_1)^2$ and $I-$ the identity.

Let for $f \in \mathbf{C}[-1,1]$ $\quad \|f\| \overset{\text{def}}{=} \max\{|f(x)| : \ -1 \le x \le 1\}$.

For a normed space $X = \mathbf{C}[-1,1]$ and a subspace Y of X induced by the operator H and given by

$$Y = \{f \in X : Hf \in X\}$$

we define the Peetre K-functional for $f \in X$ by

$$\mathbf{K}\left(f, \frac{1}{n^2}\right) = \inf_{g\in Y}\left\{\|f-g\| + \frac{1}{n^2}\|Hg\|\right\}.$$

Apart from the basic space X we define also the space $Z \subset Y \subset X$

$$Z = \left\{f \in X : \left\|H^2 f\right\| < \infty\right\}.$$

For a normed space $X = \mathbf{C}[-1,1]$ and a subspace Y_1 of X induced by the operator $H(I-L)$ and given by

$$Y_1 = \{f \in X : H(I-L)f \in X\}$$

we define the related K-functional by $K^*\left(f, \frac{1}{n^2}\right) = \inf\limits_{g\in Y_1}\left\{\|f-g\| + \frac{1}{n^2}\|H(I-L)g\|\right\}$.

We will prove that

$$\|G_{s,n}f - f\| \sim K\left(f, \frac{1}{n^2}\right) \quad \text{and} \quad \left\|G^*_{s,n}f - f\right\| \sim K^*\left(f, \frac{1}{n^2}\right) \quad \text{for} \quad s \geq 2.$$

1. Operator $G_{s,n}$.

Theorem 1.1. *If $f \in \mathbf{C}[-1,1]$, then for the operator $G_{s,n}$ and Peetre K-functional $\mathbf{K}\left(f, \frac{1}{n^2}\right)$, defined as above, we have*

$$\|G_{s,n}f - f\| \sim \mathbf{K}\left(f, \frac{1}{n^2}\right), \qquad s \geq 2.$$

Proof. Let $f(\cos(\arccos x + v)) \stackrel{\text{def}}{=} f(\cos(t+v)) \stackrel{\text{def}}{=} h(t+v)$, where $\arccos x = t$. Then $H_1 f(\cos(\arccos x + v)) = -h'(t+v)$. $H_1(H_1 f(\cos(\arccos x+v))) = h'(t+v)$. For $H \stackrel{\text{def}}{=} (\ H_1)^2$ we obtain $Hf(\cos(\arccos x + v)) = h'(t+v)$. We have

$$\begin{aligned} G_{s,n}f &= \pi^{-1}\int\limits_{-\pi}^{\pi} f(\cos(\arccos x + v))c_{n,s}\left(\frac{\sin(nv/2)}{\sin(v/2)}\right)^{2s} dv \\ &= \pi^{-1}\int\limits_{-\pi}^{\pi} h(t+v)K_{s,n}(v)dv = \pi^{-1}\int\limits_{-\pi}^{\pi} h(t-v)K_{s,n}(v)dv = \pi^{-1}h * K. \end{aligned}$$

We will utilize Theorem 3.1 (see [1, p. 69]) with

$$Q_\alpha f = G_{s,n}f \equiv \widetilde{G_{s,n}}h = \pi^{-1}K * h, \quad Df = Hf = h'', \quad \Phi(f) = \left\|H^2 f\right\| = \|\ h^{(\ 4)}\ \|,$$

where $\qquad K(v) = K^1(v) = \begin{cases} c_{n,s}\left(\dfrac{\sin(nv/2)}{\sin(v/2)}^{2s}\right), & |v| \leq \pi \\ 0, & |v| > \pi \end{cases}.$

Using Lemma 6.3 (see[1, p. 80]) as

$$\pi^{-1}\int_{-\pi}^{\pi} K(v)dv = 1, \qquad \pi^{-1}\int_{-\pi}^{\pi} vK(v)dv = 0,$$

$$\pi^{-1}\int_{-\pi}^{\pi} v^2K(v)dv = c(s)n^{-2} = \beta_1\left(\frac{1}{n}\right),$$

$$\pi^{-1}\int_{-\pi}^{\pi} v^3K(v)dv = 0, \qquad \pi^{-1}\int_{-\pi}^{\pi} v^4K(v)dv = c(s)n^{-4} = \gamma_1\left(\frac{1}{n}\right)$$

we obtain for $h^{(4)} \in \mathbf{C}[-1,1]$ i.e. $f \in Z$

$$\left\| h - \frac{1}{\pi}K * h + \frac{\beta_1(\frac{1}{n})}{2!}h' \right\| \leq \frac{\gamma_1(\frac{1}{n})}{4!}\left\| h^{(4)} \right\|. \tag{1.1}$$

This inequality will serve for (3.4) (see[1, p. 69, Th. 3.1]) with

$$\lambda_1\left(\frac{1}{n}\right) = \frac{\gamma_1(\frac{1}{n})}{4!} = O(n^{-4}), \qquad \lambda\left(\frac{1}{n}\right) = \frac{\beta_1(\frac{1}{n})}{2!} = O(n^{-2}).$$

Let

$$G_{s,n}f \equiv \widetilde{G_{s,n}}h = \pi^{-1}\int_{-\pi}^{\pi} h(t-v)K(v)dv = \pi^{-1}h * K,$$

$$\widetilde{G_{s,n}}^{\,m}h \overset{\text{def}}{=} \frac{1}{\pi^m}\underbrace{K * K * \ldots * K}_{m}, \qquad K^m \overset{\text{def}}{=} K * K^{m-1}.$$

Assertion 1.2.

$$\frac{1}{n}\left\| H_1G^m_{s,n}f \right\| \leq \text{const}\frac{1}{\sqrt{m}}\left\| f \right\|.$$

This estimation is like (6.10) (see [1, p. 82]) with $\delta = m^{-\frac{1}{2}}$ and implies

$$\left\| HG^{2m}_{s,n}f \right\| = \left\| (H_1G^m_{s,n})(H_1G^m_{s,n}f) \right\| \leq cnm^{-\frac{1}{2}}\left\| H_1G^m_{s,n}f \right\| \leq cn^2m^{-1}\left\| f \right\|. \tag{1.2}$$

This will serve for (3.6) (see [1, p. 69, Th. 3.1]).

We also have

$$\left\| H^2G^k_{s,n}f \right\| \leq cn^2m^{-1}\left\| HG^{2m}_{s,n}f \right\|, \quad \text{for} \quad k \geq 4m. \tag{1.3}$$

The inequality (1.3) will serve for (3.5) (see [1, p. 69, Th. 3.1]).

We have (3.3) (see [1, p. 69, Th. 3.1]) with $M = 1$.

To match the conditions of Theorem 4.1 (see [1, p. 72) it is sufficient to show that $A\dfrac{\lambda(n)}{\lambda_1(n)} = c\dfrac{n^2}{m}$ is satisfied with $A < 1$ i.e. $A\dfrac{O(n^{-2})}{O(n^{-4})} = c\dfrac{n^2}{m}$ is satisfied with $A < 1$ i.e

$$A = cm^{-1} < 1 \tag{1.4}$$

which is true for big m.

From (1.1), (1.2), (1.3) and (1.4) we obtain that (see [1, p. 72, Th. 4.1])

$$K\left(f, \frac{1}{n^2}\right) = \inf_g \left\{\|f-g\| + \frac{1}{n^2}\|Hg\|\right\} \le c\,\|G_{s,n}f - f\|.$$

To obtain the inequality in the opposite direction (see [1, p. 72, Th. 3.4]), we apply the following assertion.

Assertion 1.3.

$$\|G_{s,n}f - f\| \le cn^{-2}\|Hf\|.$$

Proof. Expanding $h(t+v)$ by Taylor's formula we have

$$h(t+v) - h(t) = v.h'(t) + \frac{1}{2}\int_t^{t+v} h''(s)(t+v-s)ds,$$

hence,

$$G_{s,n}f - f = \pi^{-1}\int_{-\pi}^{\pi} h(t+v) - h(t))K_{s,n}(v)dv = \pi^{-1}\frac{1}{2}\int_{-\pi}^{\pi}\int_t^{t+v} h''(s)(t+v-s)dsK_{s,n}(v)dv.$$

Then

$$\left\|\int_{-\pi}^{\pi}\int_t^{t+v} h''(s)(t+v-s)dsK_{s,n}(v)dv\right\| \le \|h''(s)\|\left|\int_{-\pi}^{\pi}\int_t^{t+v}(t+v-s)dsK_{s,n}(v)dv\right|.$$

But

$$\int_{-\pi}^{\pi}\int_t^{t+v}(t+v-s)dsK_{s,n}(v)dv = \int_{-\pi}^{\pi}\left((t+v)s - \frac{s^2}{2}\right)\Big|_t^{t+v} K(v)dv$$

$$= \int_{-\pi}^{\pi}\left[\left((t+v)^2 - \frac{(t+v)^2}{2}\right) - \left((t+v)t - \frac{t^2}{2}\right)\right]K(v)dv$$

$$= \int_{-\pi}^{\pi}\frac{v^2}{2}K(v)dv = c(s)n^{-2} = \beta_1\left(\frac{1}{n}\right) = \lambda\left(\frac{1}{n}\right), \qquad s \ge 2.$$

This concludes the proof of Theorem 1.1. $\square$

2. Operator $\mathbf{G^*_{s,n}}$. In contrast to the operator $G_{s,n}$, the operator $G^*_{s,n}$ is not positive.

Note that $G^*_{s,n} - I = (G_{s,n} - I)(I - L)$.

Theorem 2.1 *If $f \in \mathbf{C}[-1,1]$, then for the operator $G^*_{s,n}$ and Peetre K-functional $K^*\left(f, \frac{1}{n^2}\right)$ given as above we have*

$$\left\|G^*_{s,n}f - f\right\| \sim K^*\left(f, \frac{1}{n^2}\right), \qquad s \ge 2.$$

Proof. We have

$$\left\|G^*_{s,n}f - f\right\| = \|(G_{s,n} - I)(I - L)f\|.$$

Recall that $\|(G_{s,n} - I)\,F\| \sim K\left(F, \dfrac{1}{n^2}\right)$ (see Theorem 1.1). Hence

$$\left\|G^*_{s,n}f - f\right\| \sim K((I-L)f, \frac{1}{n}) = \inf_{g}\left\{\|(I-L)f - g\| + \frac{1}{n^2}\|Hg\|\right\}. \tag{2.1}$$

If we choose $g = (I - L)g_1$ we obtain

$$\begin{aligned}\left\|G^*_{s,n}f - f\right\| &\le c\inf_{g_1}\left\{\|(I-L)(f-g_1)\| + \frac{1}{n^2}\|H(I-L)g_1\|\right\}\\ &\le 2c\inf\left\{\|f-g_1\| + \frac{1}{n^2}\|H(I-L)g_1\|\right\} = cK^*\left(f, \frac{1}{n^2}\right)\end{aligned} \tag{2.2}$$

Observing that $H(ax+b) = -ax$ i.e. $H(ax+b) = -(ax+b)+b$, we have $HLg(x) = -(Lg)(x) + (Lg)(0)$.

Choosing $g = g_1 - Lf$ in (2.1), we have

$$\begin{aligned}\left\|G^*_{s,n}f - f\right\| &\sim \inf_{g_1}\left\{\|(I-L)f - g_1 + Lf\| + \frac{1}{n^2}\|Hg_1 - HLf\|\right\}\\ &= \inf_{g_1}\left\{\|f-g_1\| + \frac{1}{n^2}\|Hg_1 - HLf\|\right\}\\ &= \inf_{g_1}\left\{\|f-g_1\| + \frac{1}{n^2}\|H(I-L)g_1 - HL(f-g_1)\|\right\}\\ &= \inf_{g_1}\left\{\|f-g_1\| + \frac{1}{n^2}\|H(I-L)g_1 + L(f-g_1)(x) - (L(f-g_1))(0)\|\right\}\\ &\ge \inf_{g_1}\left\{\|f-g_1\| + \frac{1}{n^2}\|H(I-L)g_1\| - \frac{1}{n^2}\|L(f-g_1)(x) - (L(f-g_1))(0)\|\right\}.\end{aligned} \tag{2.3}$$

As the operator $L : L^\infty \to L^\infty$ is continuous, then we have

$$\|L(f-g_1)(x) - L(f-g_1)(0)\| \le c\,\|f-g_1\| \ll n^2\,\|f-g_1\|.$$

Thus the expression (2.3) for sufficiently large n is

$$\ge c\inf_{g_1}\left\{\ \|f-g_1\| + \frac{1}{n^2}\,\|H(I-L)g_1\|\right\} = cK^*\left(f, \frac{1}{n^2}\right).$$

Therefore

$$\left\|G^*_{s,n}f - f\right\| \ge cK^*\left(f, \frac{1}{n^2}\right) \tag{2.4}$$

From (2.2) and (2.4) we obtain

$$cK^*\left(f, \frac{1}{n^2}\right) \le \left\|G^*_{s,n}f - f\right\| \le cK^*\left(f, \frac{1}{n^2}\right).$$

□

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ПРИБЛИЖАВАНЕ С ОПЕРАТОРИ ОТ ТИП НА КАО-ГОНСКА $G_{S,N}$ И $G^*_{S,N}$. ПРАВИ И ОБРАТНИ ТЕОРЕМИ

Теодора Д. Запрянова

Целта на тази статия е да установи еквивалентност между ръста на приближаване с линейния оператор от типа на Као-Гонска и подходящо дефиниран K-функционал. Приложен е общ метод (развит от З. Дитциан и К. Иванов) за доказване на обратни неравенства.