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AN EXTREMAL TRANSFER THEOREM

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In this paper we investigate the asymptotic behaviour of a sequence of random indexed maxima associated with a scale-transformated point process. The initial point process consists of points whose first coordinates represent time and whose second coordinates are space. The inter arrival times are independent identically distributed random variables whose distribution tail is regularly varying with index $\beta \in (0, 1)$. The distribution function of the space points has regularly varying tail, too, with exponent $\alpha > 0$. Here is proved that such a sequence of random processes converges weakly to composition of an extremal process whose univariate marginals are Fréchet with exponent α and a hitting time process of β-stable Levy motion.

1. Introduction. There are several works on random sums, renewal counting processes and related questions. Grandel [6] deals with the total claim amount process and ruin under very general conditions. In [7] he gives an overview of the corresponding theory of Mixed Poisson processes. A recent textbook treatment of random sums is Gnedenko and Korolev [5].

This paper contains analogous result of Dobrushin's Theorem 3.1.2 in [5] for random indexed maxima. We call it Transfer Theorem, because it describes conditions providing the transfer of convergence property from the maxima of non-random number of variables to the maxima of random number.

Furrer, Michna and Weron [4] use random sums when looking at weak approximation of a risk process by α-stable Levy motion with drift. They find estimation from above for the probability of ruin in the heavy tailed case. When we tried analogously to find assessment from below we had needed theory for random indexed maxima. So we got to the idea of this paper. The application of these results to the ruin theory will appear later.

The paper is organized as follows: in §2 we remind the main concepts used further, §3 contains description of the model and in §4 the main result is formulated and proved, and some properties of the limiting process are given.

2. Preliminaries. Throughout this paper $\mathcal{M}[0, \infty)$ is the space of starting at zero, non-decreasing, right-continuous functions with finite limit on the left. All discussed here random processes have sample paths in $\mathcal{M}[0, \infty)$. We consider random variables that are defined on a given complete probability space $(\Omega, \mathcal{A}, \mathbf{P})$ with filtration $(\mathcal{A}_t)_{t\geq 0}$ and we assume that all $\mathbf{P}$ - null sets of $\mathcal{A}$ are added to $\mathcal{A}_0$. Let $\Longrightarrow$ stand for weak convergence of random processes as random elements of $\mathcal{M}[0, \infty)$, $\underset{n\to\infty}{\overset{\text{fdd}}{\longrightarrow}}$ for weak convergence of their finite dimensional distributions (fdd) and $\underset{n\to\infty}{\overset{d}{\longrightarrow}}$ for weak convergence of their one dimensional marginals. We denote by $\overset{\text{fdd}}{=}$ equality of all fdd's and by $\overset{\text{d}}{=}$ equality in distribution.

Definition 2.1. *We call the random process* $\mathbf{X} : [0,\infty) \longrightarrow [0,\infty)$ **selfsimilar with exponent** α *if its fdd satisfy the equality*

$$\mathbf{X}(st) \overset{\text{fdd}}{=} \sqrt[\alpha]{t}\mathbf{X}(s) \quad \forall t > 0. \tag{1}$$

Particularly, $\mathbf{X}(t) \overset{\text{fdd}}{=} \sqrt[\alpha]{t}\mathbf{X}(1)$.

Definition 2.2. *For every random process* $\{\mathbf{X}(t)\}_{t\geq 0}$ *in* $\mathcal{M}[0,\infty)$ *we define the* **hitting time process** *or* **first passage time process** *of* $\{\mathbf{X}(t)\}_{t\geq 0}$ *in the following way:*

$$\tau(x) = \inf\{t \geq 0 : \mathbf{X}(t) > x\} = \sup\{t \geq 0 : \mathbf{X}(t) \leq x\}$$

By Corollary 2.2.2 of [8] if $\mathbf{X}$ is selfsimilar with exponent β then τ is selfsimilar, too with exponent $1\backslash\beta$. If the process $\mathbf{X}$ is stochastically continuous then the process τ is also stochastically continuous.

Definition 2.3. *A random process* $\mathbf{X} : [0,\infty) \longrightarrow [0,\infty)$ *with the properties:*

a) $\mathbf{X}(0) \overset{\text{a.s.}}{=} 0$;

b) it has stationary increments;

c) it has independent increments;

is called a **Levy process**.

Remark: If we consider a process with additive increments, we speak on Levy process in the sum-framework. Analogously if the process has max increments, it is a Levy process in the max-framework.

Definition 2.4. *A Levy process* $\mathbf{X} : [0,\infty) \longrightarrow [0,\infty)$ *in the sum-framework, whose increments are stable random variables with index* α *is called* α**-stable Levy motion**.

Henceforth by $\{S_\beta(t)\}_{t\geq 0}$ we denote a β-stable Levy motion.

The next statement is very useful when we have to prove weak convergence of random processes. Its proof can be found in [2], Theorem 3.

Theorem 2.5. *Let* $\{\mathbf{X}_n\}_{n\in\mathbf{N}}$ *be a sequence of stochastic processes, whose path functions lie in* $\mathcal{M}[0,\infty)$. *If*

a) $\mathbf{X}_n \underset{n\to\infty}{\overset{\text{fdd}}{\longrightarrow}} \mathbf{X}$;

b) $\mathbf{X}$ *is stochastically continuous,*

then $\mathbf{X}_n \Longrightarrow \mathbf{X}$, *in the Scorohod topology.*

3. Description of the model. To make things clear, we use point pocesses. Here and below under a point process $\mathcal{N}$ we understand the collection of random points

$$\mathcal{N} = \{(J_k, X_k) : k \geq 1\}. \tag{2}$$

$J_1, J_2, \ldots$ are non-negative independent and identically distributed (iid) random variables, whose distribution function (df) G belongs to the strict domain of attraction (DA) of some stable law with index β, $\beta \in (0,1)$. This means that there exists a slowly varying function $L(n)$, such that

$$\frac{J_1+\cdots+J_n}{\sqrt[\beta]{n}L(n)} \xrightarrow[n\to\infty]{d} S_\beta,$$

where the distribution of S_β is stable with index β, briefly $G \in DA(S_\beta)$. If additionally $T_n(t) := J_1+\cdots+J_{[nt]}$ and $T_n(0) := 0$, than it is well known that there exists a regularly varying function $b(n)$ with index $\frac{1}{\beta}$, briefly $b \in RV_{\frac{1}{\beta}}$, such that

$$\frac{T_n(t)}{b(n)} \xrightarrow[n\to\infty]{\text{fdd}} \mathbf{S}_\beta(t),$$

Note that $\{\mathbf{S}_\beta(t)\}_{t\geq 0}$ is stochastically continuous with sample paths in $\mathcal{M}([0,\infty))$. We can also say that it has stationary independent incrementsit and is selfsimilar with exponent $1\backslash\beta$.

By Seneta [11] if $b(t) = \sqrt[\beta]{[t]}L([t])$, then there exists $\widetilde{b} \in RV_\beta$ such that

(3) $$b(\widetilde{b}(s)) \sim s \text{ as } s \to \infty.$$

We denote the renewal function $max\{n \geq 0 : J_1+\cdots+J_n \leq t\}$ by $\mathbf{N}(t)$ and the hitting time process of $\{S_\beta(t)\}_{t\geq 0}$ by $E(t)$. The following result, due to M. M. Meerschaert, H.P.Scheffler (Theorem 3.6 of [9]) is the main tool of this paper:

Theorem 3.1. *In the considered model*

(4) $$\frac{\mathbf{N}(nt)}{\widetilde{b}(n)} \xrightarrow[n\to\infty]{\text{fdd}} E(t).$$

Note. By the definition of $E(t)$, we can conclude that it is stochastically continuous with monotone sample paths. The path functions of the processes $\dfrac{\mathbf{N}(nt)}{\widetilde{b}(n)}$ are in $\mathcal{M}[0,\infty)$. Combining this with (4) and using Theorem 2.5, we get the convergence

(5) $$\frac{\mathbf{N}(nt)}{\widetilde{b}(n)} \Longrightarrow E(t), \quad \text{as} \quad n\to\infty \text{ in } \mathcal{M}([0,\infty)).$$

Finally let us describe the space points $\{X_n\}_{n\in\mathbf{N}}$ in (2). We suppose them iid r.v's with common *df* $F(x)$, which has the property:

$$\bar{F} \in RV_{-\alpha}, \quad \alpha > 0.$$

Further we will prove that for the time changes $\tau_n(t) = \dfrac{t}{n}$, there exist space changes $u_n(t) = \dfrac{t}{B(n)}$, where $B(n)$ are positive numbers, such that the extremal process associated with the point process

(6) $$\mathcal{N}_n = \{(\tau_n(J_k), u_n(X_k)) : k \geq 1\} \quad n \in \mathbf{N},$$

converges weakly to the composition of an extremal process Y_α with Fréchet's univariate marginals and the hitting time process $\mathbf{E}$.

4. The main result.

Theorem 4.1. *In the considered model there exists a sequence* $\{B(n)\}_{n\in\mathbf{N}}$ *such that*

$$\{\bigvee_k u_n(X_k) : \tau_n(J_k) \leq t\} = \bigvee_{k=1}^{\mathbf{N}(nt)} \frac{X_k}{B(n)} \Longrightarrow \mathbf{Y}_\alpha(\mathbf{E}(t)),$$

where $\mathbf{Y}_\alpha$ *is an extremal process with Fréchet one-dimensional marginals with exponent* α *and* $\mathbf{E}$ *is the hitting time process of the strictly stable one-sided Levy motion* $\{\mathbf{S}_\beta(t)\}_{t\geq 0}$.

Moreover: a) when $n \to \infty$

$$B(n) \sim F^{\leftarrow}\left(1 - \frac{1}{\widetilde{b}(n)}\right) \in RV_{\beta\alpha^{-1}}.$$

where $\widetilde{b} \in RV_\beta$ *is defined in (3);*

b) The processes $\mathbf{Y}_\alpha$ *and* $\mathbf{E}$ *are independent.*

Proof. $\bar{F} \in RV_{-\alpha}$. By Resnick's Invariance Principle for maxima – Theorem 4.20 of [10], if

$$Y_n(t) = \begin{cases} (\bigvee_{k=1}^{[nt]} (X_k - a_n)/b_n^* & t \geq n^{-1} \\ (X_1 - a_n)/b_n^* & 0 < t < n^{-1} \end{cases} . \tag{7}$$

then

$$Y_n(t) \Longrightarrow \mathbf{Y}_\alpha(t) \text{ in } \mathcal{M}[0, \infty) \tag{8}$$

and $\{\mathbf{Y}_\alpha(t)\}_{t\geq 0}$ is an extremal process generated by Fréchet univariate marginals. Furthermore $a_n \sim 0$ and

$$b_n^* \sim F^{\leftarrow}\left(1 - \frac{1}{n}\right) \in RV_{\alpha^{-1}}. \tag{9}$$

Since $\widetilde{b}(n)$ converges to infinity when $n \to \infty$, then

$$Y_{\widetilde{b}(n)}(t) \Longrightarrow \mathbf{Y}_\alpha(t) \text{ in } \mathcal{M}[0, \infty). \tag{10}$$

Note that J_k and X_k are independent. The processes $\mathbf{Y}_\alpha$ and $\mathbf{N}(t)$ are independent and hence it follows from (10) together with (5) that we also have

$$\{Y_{\widetilde{b}(n)}(t), \frac{\mathbf{N}(nt)}{\widetilde{b}(n)}\} \Longrightarrow \{\mathbf{Y}_\alpha(t), \mathbf{E}(t)\} \quad \text{as} \quad n \to \infty. \tag{11}$$

in J_1-topology of Scorohod and hence also in the weak topology of $\mathcal{M}[0, \infty)$. Since the process $\{\mathbf{E}(t)\}_{n\in\mathbf{N}}$ is not strictly increasing Theorem 3.1 in Whitt [12] does not apply, so we can not prove convergence in J_1-topology. Instead we use Theorem 13.2.4 in Whitt [13], which applies as long as $x = E(t)$ is a.s. strictly increasing whenever $\mathbf{Y}_\alpha(x) \neq \mathbf{Y}_\alpha(x^-)$. This condition is easily shown to be equivalent to the statement that the independent processes $\{\mathbf{S}_\beta(t)\}$ and $\{\mathbf{Y}_\alpha(x)\}$ a.s. have no simultaneous jumps, which is easy to check. So the composition is stochastically continuous and we can apply Continuous Mapping Theorem – (see Billingsley [1] Th. 5.1 and Th. 5.5). This fact together with (7) and (10) leads to

$$\bigvee_{k=1}^{\mathbf{N}(nt)} \frac{X_k}{B(n)} \stackrel{\text{fdd}}{=} Y_{\widetilde{b}(n)}(\frac{\mathbf{N}(nt)}{\widetilde{b}(n)}) \Longrightarrow \mathbf{Y}_\alpha(\mathbf{E}(t)) \text{ as } n \to \infty. \tag{12}$$

This is the first part of our conclusion.

From (7) with $a_n \sim 0$ and (12) $B(n) = b^*_{\widetilde{b}(n)}$. Substituting (9) in the last equality we get

$$B(n) \sim F^{\leftarrow}\left(1 - \frac{1}{\widetilde{b}(n)}\right), \quad n \to \infty.$$

So we proved the asymptotic equivalence in a). Since $\bar{F} \in RV_{-\alpha}$, then $\frac{1}{\bar{\bar{F}}} \in RV_{\alpha}$ and its inverse $F^{\leftarrow}(1-\frac{1}{n})$ is regularly varying with index $\frac{1}{\alpha}$. Let us remind that $\widetilde{b}$ was RV_{β}. The function in a) is composition of $\widetilde{b}$ and $F^{\leftarrow}\left(1-\frac{1}{n}\right)$, so it is $RV_{\frac{\beta}{\alpha}}$.

b) $\{X_k\}_{k\in\mathbf{N}}$ and $\{J_s\}_{s\in\mathbf{N}}$ are independent. $\mathbf{Y}_\alpha$ depends on X and does not depend on J. Analogously, the random process $\mathbf{E}$ depends on J and does not depend on X. Two random processes are defined on the same probability space, this is why they are independent. □

Theorem 4.2. *In the considered model the limiting process* $\mathbf{Y}_\alpha \circ \mathbf{E}$ *is*

a) selfsimilar with exponent $\beta\alpha^{-1}$ *i.e.*

$$\mathbf{Y}_\alpha \circ \mathbf{E}(st) \stackrel{\text{fdd}}{=} \sqrt[\alpha]{s}^{\beta}\mathbf{Y}_\alpha \circ \mathbf{E}(t) \tag{13}$$

and $\mathbf{Y}_\alpha \circ \mathbf{E}(t) \stackrel{\text{fdd}}{=} \sqrt[\alpha]{t}^{\beta}\mathbf{Y}_\alpha \circ \mathbf{E}(1) \stackrel{\text{fdd}}{=} \sqrt[\alpha]{t}^{\beta}\mathbf{Y}_\alpha(\mathbf{S}_\beta^{-\beta})$*;*

b) stochastically continuous.

c) For all $x \geq 0$ *its one dimensional marginals are*

$$\mathbf{P}(\mathbf{Y}_\alpha \circ \mathbf{E}(t) < x) = \sum_{n=0}^{\infty} \frac{(-x^{-\alpha}t^{\beta})^n}{\Gamma(1+n\beta)}, \quad \beta \in (0,1);$$

Proof. a) By the definitions of $\{\mathbf{N}(t)\}$ and $\{J_n\}$ and by the note after Theorem 3.1 we know that (5) is fulfilled and $\{\mathbf{E}(t)\}_{t\geq 0}$ is the hitting time process of the strictly β-stable Levy motion $\{\mathbf{S}_\beta(t)\}_{t\geq 0}$. By Proposition 3.1 of [8] we know that $\{\mathbf{E}(t)\}_{t\geq 0}$ is a selfsimilar process with exponent β. Note that $\{\mathbf{Y}_\alpha(t)\}_{t\geq 0}$ is selfsimilar with exponent $1/\alpha$. So, because the composition of selfsimilar processes is again a selfsimilar process with exponent the product of the both exponents we get that $\mathbf{Y}_\alpha \circ \mathbf{E}$ is selfsimilar with exponent $\beta\alpha^{-1}$. Now (13) follows by (1).

b) As a composition of two stochastically continuous processes $\mathbf{Y}_\alpha \circ \mathbf{E}$ is stochastically continuous.

c)
$$\begin{aligned}
\mathbf{P}(\mathbf{Y}_\alpha \circ \mathbf{E}(t) < x) &= \mathbf{P}(\mathbf{Y}_\alpha \circ \mathbf{E}(1) < \frac{x}{\sqrt[\alpha]{t^\beta}}) = \\
&= \int_0^\infty \mathbf{P}(\mathbf{Y}_\alpha(z) < \frac{x}{\sqrt[\alpha]{t^\beta}}) d\mathbf{P}(\mathbf{E}(1) < z) = \\
&= \int_0^\infty \exp\{-z(\frac{x}{\sqrt[\alpha]{t}^{\beta}})^{-\alpha}\} d\mathbf{P}(\mathbf{E}(1) < z) = \\
&= \int_0^\infty \exp\{-zt^\beta x^{-\alpha}\} d\mathbf{P}(\mathbf{E}(1) < z) = \\
&= \mathbf{E}\exp\{-t^\beta x^{-\alpha}\mathbf{E}(1)\} = \mathbf{E}\exp\{-t^\beta x^{-\alpha}(\mathbf{S}_\beta)^{-\beta}\},
\end{aligned}$$

where the last equality follows by Corrolary 3.2,(a) in [9].

In [3] is shown that $(S_\beta)^{-\beta}$ is Mittag-Leffler distributed. So, we completed the proof. □

Remarks. 1. In c) for $\beta = 0$ we get the exponential law and for $\beta = 1$ we get the degenerate law.

2. When $\alpha = \beta$ the sample paths of the composition of $\mathbf{Y}_\alpha$ and $\mathbf{E}$ are straight lines with random slope.

REFERENCES

[1] P. Billingsley. Convergence of Probability Measures, Wiley, New York, 1968.

[2] N. H. Bingham. Limit Theorems for Occupation Times of Markov Processes. *Z. Wahrsch. verw. Geb.*, **17**, 1–22.

[3] L. Bondesson, G. Kristiansen, F. Steutel. Infinite Divisibility of Random Variables and Their Integer Parts. *Statist. Probab. Lett.*, **28** (1996), 271–278.

[4] H. Furrer, Z. Michna, A. Weron. Stable Levy Motion Appriximation in Collective Risk Theory, Insurance. *Mathematics and Economics*, **20** (1997), 97–114.

[5] B. V. Gnedenko, V. Yu. Korolev. Random Summation, CRC Press, Boca Raton, 1996.

[6] J. Grandell. Aspects of Risk Theory. Springer, Berlin, 1991.

[7] J. Grandell, Mixed Poisson Processes, Chapman and Hall, London, 1997.

[8] P. K. Jordanova. On Self Similar Random Processes with Non-Increasing Sample Paths, University of Russe, 2003 (to appear).

[9] M. M. Meerschaert, H. P. Scheffler. Limit Theorems for Continuous Time Random Walks, 2002 (to appear).

[10] S. I. Resnick. Extreme Values, Regular Variation and Point Processes, Springer, New York, 1987.

[11] E. Seneta. Regularly Varying Functions. Lecture Notes in Mathematics, vol. **508**, Springer-Verlag, Berlin, 1976.

[12] W. Whitt. Some Useful Functions for Functional Limit Theorems. *Mathematics of Operations Research*, **5** (1980), 67–85.

[13] W. Whitt. Stochastic Process Limits. Springer, New York, 2002.

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ЕДНА ЕКСТРЕМАЛНА ТЕОРЕМА ЗА ПРЕНОСА

Павлина К. Йорданова

В статията се изследва асимптотичното поведение на редица от случайно индексирани максимуми, асоциирани с мащабно трансформиран точков процес. Изходният точков процес се състои от точки, чиито първи координати представляват времето, а вторите – пространството. Моментите от време са независими, еднакво разпределени случайни величини, чиято функция на разпределение има правилно изменяща се опашка, но с индекс $\beta \in (0, 1)$. В статията доказваме, че една такава редица от случайни процеси е слабо сходяща към композиция на екстремален процес, генериран от едномерни маргинали на Фреше с експонента и момента на достигане на дадено ниво от β-устойчиво движение на Леви.