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THE DIACHRONIC TEACHING OF GEOMETRY AND ITS CONTRIBUTION TO THE DEVELOPMENT OF SKILLS

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In primary and also very often in secondary education the proofs of various proposals cannot have a clearly formal character as the students have not yet reached that level, which would allow them such approach. For this reason, proofs should have many instinctive elements. The paper considers such proofs in connection with the so-called basic geometrical proposals: equality of two triangles, the sum of the angles of the triangle, Thales' theorem, Pythagorean theorem and the theorem of Ptolemy.

Most historians of Mathematics agree on the fact that the origins of Geometry are strongly connected with the need of description of certain objects, activities and natural phenomena.

The sun and the stars could indicate a prime view about what the concepts of circular disk and point could mean. The way the rain and bodies fall, could constitute an aspect of the perpendicular (line) while the surface of the lake – the plane. Similarly, waves could give us an aspect of the curve and periodical movement, as well as the dandling of the grass by the wind and the move or the orbit of the moving staircase.

In addition, the fruit, the shell and the pebble etc could indicate a view of some concepts such as shape, unit, equality, inequality etc. Following this pattern, all the concepts of Euclidean Geometry were defined and presented not only in Euclid "Elements", but also in Euclidean Geometry, as presented today.

The comprehension of these basic concepts allow the realisation of a series of manufactures as residences, bunkers, temples and several worship buildings and objects.

The same period, around 3000 B.C, the first written monuments were presented. Therefore, our information about people's awareness at that time as far as their Knowledge of Geometry is concerned, are in a satisfactory degree precise.

To give an example, written texts of the people of Mesopotamia and Egypt can inform us about the kind of practical geometry these people had developed years ago. Effective for their daily needs but different from generative science, this kind of Geometry did not use the concept of proof, which was discovered much later (around 600 B.C.) by Thales of Miletus and was developed by Pythagoreans.

The need for bigger and safer buildings contributed in the need for more complete knowledge about issues of Geometry dealing with concepts such as verticality, parallelism and metrics Geometry that are generated by Pythagoras and his students aiming at the numerisation of knowledge of Mathematics as a whole. The so called Pythagorean theorem, beyond any doubt, has a significant scientific value as it constitutes the most applicable theorem and the main basis of metrics Geometry. Findings from ancient years has rendered known that in the year 3000 B.C., in Mesopotamia there were schools where students were taught Practical Geometry.

The teaching of Geometry, in any form, appears in ancient Mesopotamia, Egypt, and Greece and becomes more methodical in Thales (600 B.C.) and Pythagorean times.

It becomes, therefore, obvious that the diachronic teaching of Geometry in Primary and Secondary education is considered necessary, as it contributes not only in the development of essential skills but also in the comprehension/understanding of a wide spectrum of mathematical concepts and concepts from other scientific fields.

Besides, it is known that all the ancient Greek philosophers believed in the instructive value of Geometry. For example, Plato thought knowledge of Geometry was necessary condition in order students to deal with matters of Philosophy.

In addition to the above, the development of Mathematics and especially Geometry is to pure democracy that the citizens would experience in ancient Greece. Of course, pure democracy required the development of dialogue and reasonable arguments, which people could develop mainly through the knowledge of mathematic structures.

The first proofs given in geometrical proposals by Thales, Pythagoreans or other posteriors up to the time of Plato have not been rescued. In the Platonic dialogue "Menon" Socrates uses the maieutic method in order to guide the young slave in the solution of the problem of "*given rectangle to be drawn rectangle with area double than the area given*". The proof Plato reports (which the slave is supposed to have given) belongs to an anterior mathematician or in Plato himself and is the following:

Consider the rectangle ΑΒΓΔ (Fig. 1), if we draw the intersecting diagonals in O point ΑΓ, ΒΔ creating the rectangle ΕΖΗΘ we resolve the problem, as the main rectangle ΑΒΓΔ, consists of 4 equal right triangles, and the ΕΖΗΘ triangle of 8 such triangles. The obvious proof of Plato gives the impression that even axioms are not required (as Philosopher Popper points out in [10]), however obviously this is not true, as the equality of the four triangles is justified only with the use of an axiomatic system.

In any case, it is obvious that the equality of two plane figures even in "Elements" is based on the instinctive method of attack (Proposal I.4). This method is supported substantially by the fact that a plane figure remains invariant on a move which is applied on another given, in order to compare them.

This obviously happens when the figure is concrete, i.e. when its two sides are stable as the containing angle which they shape. In essence, the method of attack is based on the axiom ΠΓΠ, and as a result, overlooking its strictness, the practical conclusions of such a proof are acceptable. With this method, for instance, it is likely that the ancient Eastern people are those who have approached the so called theorem of Pythagoras.

Euclid in his first book of "Elements" gave a monumental proof of Pythagorean Theorem with the division of the figure in equal and equivalent triangles. However, in the second book of Euclid's "Elements" we can also find the proposal $2\beta\gamma + (\beta - \gamma)^2 = \alpha^2$ (Proposal II.7), where α is the hypotenuse and β, γ the vertical sides of the right triangle ΑΒΓ. In this way we are able to give an "obvious" proof of Pythagorean Theorem as shown in Figure 2, where one square ΑΒΓΔ of side α has been divided in four equal right triangles ΑΒΖ, ΒΓΗ, ΓΔΘ, ΑΔΕ and in one rectangle ΕΖΗΘ. If we consider that ΑΖ=β, and ΒΖ = γ then we have $\alpha^2 = 4$ **Error!** $\beta\gamma + (\beta - \gamma)^2$ or $\alpha^2 = \beta^2 + \gamma^2$, that is to say the Pythagorean theorem.

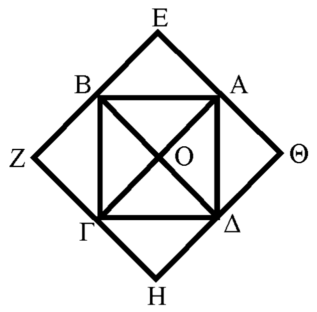

Fig. 1

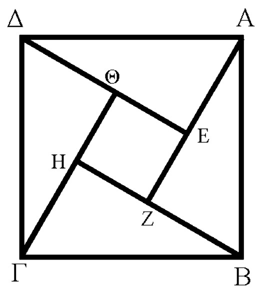

Fig. 2

Perhaps, this proof belongs either to Pythagoreans or to a certain Greek mathematician prior to Euclid. The two examples above constitute proofs of proposals with strong instictive elements that, even if they are not distinguished for their strictness, however are mathematically precise and above all easy to comprehend by students with limited knowledge of mathematics. Moreover,it is admirable that we are able to comprehend and describe the real space we live in with the help of simple geometrical objects, which according to the model that Euclid first applied in “Elements”, we distinguish in initial concepts, common concepts (axioms), proposals.

In this respect, Euclidean Geometry constitutes an insuperable in perfection model of Mathematical theory. Since 600 B.C., when Thales introduced the method of proving up to the 300 B.C. when Euclid introduced the axiomatic foundation, Geometry took its definite form up today, where the proofs of proposals are independent of the figure and intuition that even Plato himself had considered necessary.

Indeed, in Primary as well as in Secondary education the proofs of various proposals can not have a clearly formal character, as the students have not yet reached that van Hiele level, which would allow them such an approach.

For this reason, proofs should have many instictive elements and for strictly pedagogical aims they should accompanied with historical real problems. In fact, the formulation of the proposals rised in order to deal with these problems.

Although this process is impossible to be applied in each geometrical proposal, mainly for practical reasons however, it could be applied in certain basic proposals.

The characterization “basic proposals” is referred to the fact that such a proposal is of high importance for the structure of Euclidean Geometry itself or the ammount of possible applications the proposal could have.

Thus, taking under consideration the above criteria we could define as basic geometrical proposals the following:

(i) **Equality of two triangles (one of the ΠΓΠ, ΓΠΓ, ΠΠΠ proposals) and historical applications.** From a historical aspect, we know that Thales had used the equality of two triangles in order to calculate the distance between two not approachable points, e.g. the distance of a boat from the harbor etc. In “Elements” Euclid mentions and uses the proposals ΠΓΠ, ΓΠΓ, ΠΠΠ in book I.

(ii) **The sum of the angles of the triangle (and each convex polygon).** The proposal that the sum of the angles of the triangle is equal to 180° is known as equivalent with the parallels axiom of Euclid and it is useful not only for the solution of many real problems (e.g. Topography), but also for the comprehension that this proposal is valid only in Euclid's level and not in a spherical triangle. A brilliant "practical" proof of this proposal recommended for the study of the proposal in obligatory education is the following:

Consider a triangle ΑΒΓ (Figure. 3) and in point A, a person moving along its sides in the direction $\mathrm{B} \rightarrow \mathrm{A} \rightarrow \Gamma \rightarrow \mathrm{B}$. If so, then he will draw/construct $180° - \mathrm{B}$, $180° - \mathrm{A}$, $180° - \Gamma$ while in his return in point B will have drawn 360°, consequently $180° - \mathrm{A} + 180° - \mathrm{B} + 180° - \Gamma = 360° \Rightarrow +\mathrm{B} + \Gamma = 180°$. In addition to Euclidean elements, this proof contains not Euclidean ones too. Such elements are the concept of "direction", which the students will analytically study in the Lyceum.

A kind of implementation of this proposal could be the introduction of the "$\alpha\nu\alpha\lambda\lambda o\omega\tau\eta$" regarding the sum of the external angles of a convex n-angle.

(iii) **Thales' theorem (in its general form), where each pair of (parallel or not) beelines is divided by parallel lines in proportional parts** (Fig. 4). The importance of Thales' theorem is essential, not only for the development of homotheticity, the resemblance in the plane or the 3-dimensional space and the theorems of bundle, bisector, and congruous reasons but also for the development of Metrical Geometry. Moreover, Thales used this proposal, as many historians report, in a series of applications such as the calculation of the height of the pyramid considering the length of its shade at a certain time, the drawing of maps of various small regions, astronomy etc.

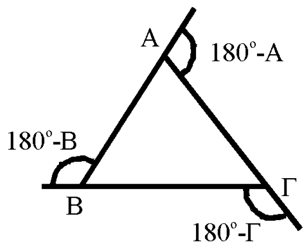

Fig. 3

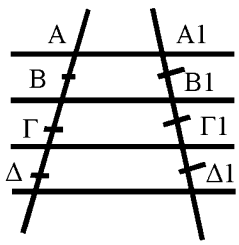

Fig. 4

A teaching perspective of the theorem in the general form is exceptionally difficult and even though it requires knowledge gained and developed after Thales times, mainly by Eudoxus and Theaitetus, even today is difficult. However, the proposal could be approached "theoretically" in compulsory education based on the partial proposal of Thales, where a part is divided in equal parts by parallel lines, and therefore it will also divide each part equally.

(iv) Due to the fact that the **Pythagorean Theorem** is equivalent to the axiom of parallels, it can not be applied in a "right" but not Euclidean triangle. This theorem has given rise to the discovery of irrational numbers by Pythagorean Ippasus. Additionally, one Pythagorean, Theodore of Cyrenia studied the irrational numbers and constructed a non symmetrical number (the square root of v), where ν belongs to N^*, as the half of a beeline of a right triangle with hypotenuse n+1 and other beeline n-1. According to historians of Mathematics, the Pythagorean Theorem may be the most important one of Euclidean Geometry and constitutes the basic proposal of metric Euclidean Geometry. Obviously, it is equivalent with the basic geometrical congruence $\sin^2 \omega + cos^2\omega = 1$ that is valid for each ω angle. The above theorem contributed on the introduction of the concept of co-ordinate initially by Apollonius with Descartes following and the concept of metrics in Euclidean and not abstract metric spaces. Moreover, the applications of the theorem in Geometry and in various mathematical fields are innumerous/infinite.

(v) A teaching approach of the theorem in Primary education could be via Figure 5 where a rectangle ΑΒΓΔ (Fig. 5) is divided by its diagonals ΑΓ, ΒΔ in four right triangles ΑΟΒ, ΒΟΓ, ΓΟΔ, and ΑΟΔ. If we move ΑΟΒ, ΓΟΔ in positions ΒΓΖ, ΑΔΕ we notice that the sum of the area of the rectangle with sides ΟΓ, ΟΔ equals to the area of the rectangle with its side the hypotenuse ΓΔ of the right triangle ΓΟΔ.

(vi) According to the **theorem of Ptolemy**, for each drawn quadrilateral figure ΑΒΓΔ (Fig. 6) the following relation is valid:

$$(i) \qquad \mathrm{A\Gamma} \cdot \mathrm{B\Delta} = \mathrm{AB} \cdot \Gamma\Delta + \mathrm{A}\Delta \cdot \mathrm{B}\Gamma.$$

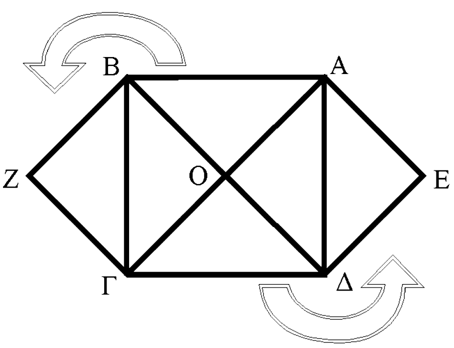

Fig. 5

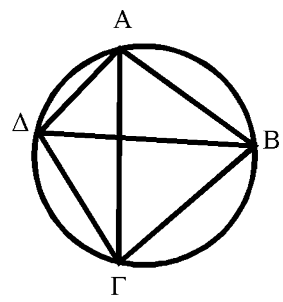

Fig. 6

Euclid was aware of this theorem, although Ptolemy used it first as generalization of the Pythagorean Theorem. Indeed, if ΑΒΓΔ is a rectangle and therefore $\mathrm{AB} = \Gamma\Delta$, $\mathrm{A}\Delta = \mathrm{B}\Gamma$, then $\mathrm{A}\Gamma = \mathrm{B}\Delta$ and (i) must be $\mathrm{AB}^2 + \mathrm{B}\Gamma^2 = \mathrm{A}\Gamma^2$. This theorem was used by Ptolemy in order to construct and prove common relations in his geometrical system such as:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

In Euclidean complex plane the theorem of Ptolemy can receive the following form:

$$|Z_1 - Z_2|\,|Z_3 - Z_4| + |Z_1 - Z_4|\,|Z_@ - Z_3| = |Z_1 - Z_3|\,|Z_2 - Z_4|,$$

where Z_1, Z_2, Z_3, Z_4 are representations of Α, Β, Γ, Δ.

Apart from the proposals mentioned above , attributes/qualities as the principle of "duality" could be of major educational importance. Based on this principle, we could have a better approach on theorems of affine (Menelaus-Céva etc.) and on mathematical induction.

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ДИАХРОНИЧНОТО ПРЕПОДАВАНЕ НА ГЕОМЕТРИЯ И НЕГОВИЯТ ПРИНОС В РАЗВИВАНЕТО НА УМЕНИЯ

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В началото, а също доста често и в средното образование доказателствата на различни твърдения не могат да имат чисто формален характер, тъй като учениците все още не са достигнали онова ниво, което би позволило такъв подход. Поради тази причина доказателствата трябва да съдържат доста инстинктивни елементи. Статията разглежда такива доказателства във връзка с т.нар. основни геометрични твърдения: еднаквост на два триъгълника, сбор на ъглите в триъгълника, теоремата на Талес, Питагоровата теорема и теоремата на Птоломей.