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ASYMPTOTIC STABILIZATION OF A BIOTECHNOLOGICAL PROCESS WITH SUBSTRATE INHIBITION*

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A nonlinear model of methane fermentation involving substrate inhibition is studied. A continuous feedback is proposed, which stabilizes asymptotically the dynamic system towards an operating point, chosen according to a practical criterion. Numerical simulations in Maple demonstrate the theoretical results.

1. Introduction. We consider a model of methane fermentation based on two nonlinear ordinary differential equations and one algebraic nonlinear equation [2], [5],

$$\frac{ds}{dt} = -k_1\mu x + u(s_{\rm in} - s) \tag{1}$$

$$\frac{dx}{dt} = (\mu - u)x \tag{2}$$

$$Q = k_2\mu x, \tag{3}$$

where $x = x(t)$ and $s = s(t)$ are the state variables and μ is the specific biomass growth rate. This model can also be considered as describing the final (methanogenic) path of the methane fermentation [1], [2], [4], in continuously stirred tank bioreactors.

The model (1)–(3) is studied under the following assumptions:

(i) the influent substrate concentration $s_{\rm in}$ is constant and $s < s_{\rm in}$ holds true;

(ii) the dilution rate u is the control input. We assume that $u \in \mathcal{U}$, where $\mathcal{U}$ is a compact set of admissible *positive* values for the control;

(iii) the specific growth rate μ depends on the state variable s, that is $\mu = \mu(s)$.

We consider the growth rate function $\mu(s)$ to be described by the Haldane law (cf. [2])

$$\mu(s) = \frac{\mu_{\rm m} s}{k_{\rm s} + s + s^2/k_{\rm i}}.$$

The function $\mu(s)$ has a maximum at the point $\hat{s} = \sqrt{k_{\rm s}k_{\rm i}}$. This fact exhibits the so called substrate inhibition phenomenon representing a feedback in the model [2], [5].

The definition of the model variables and parameters is listed in Table 1.

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Table 1:

	Model variables and parameters	Values	Units
x	biomass concentration	–	g/dm^3
s	substrate concentration	–	g/dm^3
u	dilution rate	–	day^{-1}
s_{in}	influent substrate concentration	2	g/dm^3
k_1	yield coefficient	3	–
k_2	coefficient	5.6	$(dm^3)^2/g$
Q	methane gas flow rate	–	dm^3/day
μ_{m}	maximum specific growth rate	0.35	day^{-1}
k_{s}	saturation constant	0.7	g/dm^3
k_{i}	inhibition coefficient	0.6	g/dm^3

The paper is organized as follows. In Section 2 we compute the optimal static point with respect to a given practical criterion. In Section 3 we propose a continuous feedback, which stabilizes asymptotically the dynamic process to the optimal static point. Section 4 reports on computer simulations in Maple.

2. The optimal static point. The steady states model of the process is obtained from (1)–(2) by setting $ds/dt = 0$ and $dx/dt = 0$. Excluding the trivial solutions $s = 0$ or $s = s_{\text{in}}$ and $x = 0$ (which are called washout steady states and are not of practical interest) it is straightforward to check that for any u from the interval

$$U = \left(0,\ \frac{\mu_{\text{m}}}{1 + 2\sqrt{k_{\text{s}}/k_{\text{i}}}}\right]$$

there exists an unique stable steady state $(s(u), x(u))$ (cf. [2]) with

$$s(u) = \frac{k_{\text{i}}}{2}\left(\frac{\mu_{\text{m}}}{u} - 1 - \sqrt{\left(\frac{\mu_{\text{m}}}{u} - 1\right)^2 - 4\frac{k_{\text{s}}}{k_{\text{i}}}}\right), \quad x(u) = \frac{s_{\text{in}} - s(u)}{k_1}.$$

Moreover, every steady state $(s(u), x(u))$ belongs to the line segment

$$H = \{(s, x) :\ s + k_1 x = s_{\text{in}},\ 0 \le s \le s_{\text{in}}\},$$

which is strongly invariant with respect to the trajectories of (1)–(2) (cf. [3], p. 198), i. e. every trajectory of (1)–(2) starting from a point of H remains in H.

After substituting $\mu(s) = u$ and $x = x(u)$ in (3), the output Q is obtained as function of the control input u, that is

$$Q(u) = k_2 u x(u).$$

$Q(u)$ is called input-output static characteristic of the dynamic process. There exists a unique point u^* where $Q(u)$ takes its maximum, i. e. $Q(u^*) = \max_{u \in U} Q(u)$. It can be directly verified that

$$u^* = \frac{\mu_{\text{m}}}{1 - 4k_{\text{s}}/k_{\text{i}}}\left(1 - \frac{1 + 2s_{\text{in}}/k_{\text{i}}}{\sqrt{1 + (s_{\text{in}}/k_{\text{s}})(1 + s_{\text{in}}/k_{\text{i}})}}\right).$$

Denote

$$s^* = s(u^*), \quad x^* = x(u^*).$$

The point (s^*, x^*) is called optimal static point and obviously it belongs to the invariant set H.

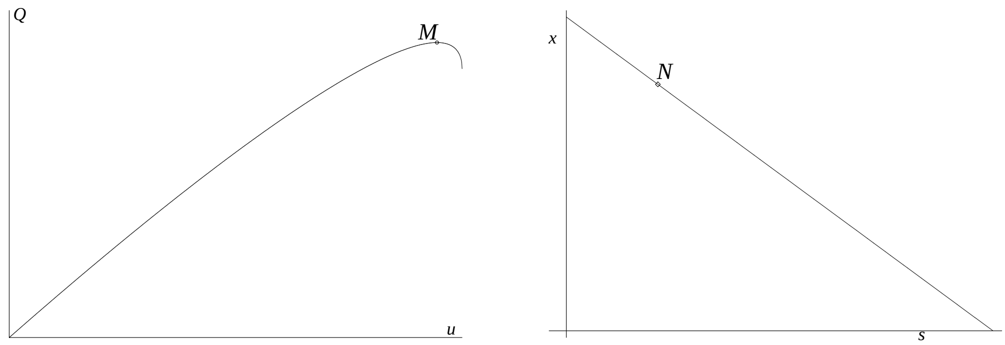

Figure 1: The input-output static characteristic $Q(u)$ (left) and the invariant set H with the optimal static point N (right)

Figure 1 presents the input-output static characteristic $Q(u)$ with the maximum point $M = (u^*, Q(u^*))$ (left plot) and the invariant segment H with the point $N = (s^*, x^*)$ (right plot).

3. The stabilizing feedback. We shall construct a continuous feedback, stabilizing asymptotically the dynamic system (1)–(2) to the optimal static point (s^*, x^*).

The smooth change of the state variables

$$\xi = \frac{x - x^* - k_1(s - s^*)}{1 + k_1^2}, \quad \eta = \frac{s - s^* + k_1(x - x^*)}{1 + k_1^2} \tag{4}$$

transforms the system (1)–(2) into a simpler and more convenient one,

$$\begin{aligned} \frac{d\xi}{dt} &= f(\xi, \eta; u) \\ \frac{d\eta}{dt} &= -u\eta \end{aligned} \tag{5}$$

with

$$f(\xi, \eta; u) = \frac{\mu_{\mathrm{m}}(s^* - k_1\xi + \eta)(x^* + \xi + k_1\eta)}{k_{\mathrm{s}} + s^* - k_1\xi + \eta + 1/k_{\mathrm{i}}(s^* - k_1\xi + \eta)^2} - u \cdot (x^* + \xi). \tag{6}$$

Obviously, the point (s^*, x^*) is mapped into the origin $O = (0, 0)$ by the coordinate change (4).

Denote by B_r the closed disc in R^2 with radius $r > 0$ and center O. Remind that by $\mathcal{U}$ we have denoted the set of admissible *positive* values for the control.

Definition 1. *Every continuous function $k : B_r \to \mathcal{U}$ is called a continuous feedback. The feedback $k : B_r \to \mathcal{U}$ is said to stabilize asymptotically the system (5) to the origin if*

(a) for every point $(\xi_0, \eta_0) \in B_r$ the solution $(\xi(t), \eta(t))$ of (5), starting from (ξ_0, η_0) and corresponding to the feedback k, is defined on $[0, +\infty)$ and remains in B_r;

(b) $(\xi(t), \eta(t))$ tends to the origin as $t \to +\infty$.

Definition 2. *The control system (5) is said to be locally asymptotic stabilizable to the origin if there exists a radius $r > 0$ and a continuous feedback $k : B_r \to \mathcal{U}$, such that k stabilizes asymptotically the system to the origin.*

The property asymptotic stabilizability does not depend on the choice of the coordinate axes, thus we shall look for a feedback, stabilizing asymptotically the control system (5) to the origin.

We set

$$\hat{u}(\xi,\eta) := \frac{\mu_{\mathrm{m}}(s^* - k_1\xi + \eta)(x^* + \xi + k_1\eta)}{(k_{\mathrm{s}} + s^* - k_1\xi + \eta + 1/k_{\mathrm{i}}(s^* - k_1\xi + \eta)^2)(x^* + \xi)}. \tag{7}$$

Since $\hat{u}$ is a continuous function and $\hat{u}(0,0) > 0$, there exists $r > 0$ such that the values of $\hat{u}$ on the disc B_r are positive. Let be $\delta > 0$. Define further

$$u_{\min}(\delta, r) = \min_{(\xi,\eta)\in B_r} \hat{u}(\xi,\eta) + \delta\xi, \quad u_{\max}(\delta, r) = \max_{(\xi,\eta)\in B_r} \hat{u}(\xi,\eta) + \delta\xi.$$

Now we can formulate the main result.

Proposition. *Let there exist $\delta > 0$ and $r > 0$ such that*

$$U \cup \mathcal{I} \subseteq \mathcal{U},$$

where $\mathcal{I} = [u_{\min}(\delta, r), u_{\max}(\delta, r)]$. Then the feedback

$$k(\xi,\eta) = \hat{u}(\xi,\eta) + \delta\xi \tag{8}$$

is a continuous admissible control function defined on B_r which stabilizes asymptotically the control system (5) to the origin $(0,0)$.

Proof. By substituting $u = k(\xi,\eta)$ in (5) we obtain the system

$$\begin{aligned} \frac{d\xi}{dt} &= -\delta\xi \\ \frac{d\eta}{dt} &= -k(\xi,\eta)\eta. \end{aligned} \tag{9}$$

Since $k(\xi,\eta) \geq u_{\min}(\delta, r) > 0$ for every point $(\xi,\eta) \in B_r$, one can easily check that $w(\xi,\eta) = \xi^2 + \eta^2$ is a Lyapounov function for the system (9). Indeed,

$$\begin{aligned} \frac{dw}{dt} &= 2\xi\frac{d\xi}{dt} + 2\eta\frac{d\eta}{dt} = -2\delta\xi^2 - 2\eta^2 \cdot k(\xi,\eta) \\ &\leq -2\delta\xi^2 - 2u_{\min}(\delta, r)\eta^2 \begin{cases} < 0 & \text{for } (\xi,\eta) \neq 0, \\ = 0 & \text{for } (\xi,\eta) = 0. \end{cases} \end{aligned}$$

Applying Theorem 5.5 from [3], it follows that the control system (5) is asymptotically stabilizable to the origin. □

Remark. In (s,x)-coordinates the feedback (8) takes the form

$$k(s,x) = (1 + k_1^2)\frac{\mu(s)\cdot x}{k_1(s_{\mathrm{in}} - s) + x} + \delta\frac{x - x^* - k_1(s - s^*)}{1 + k_1^2} \tag{10}$$

and stabilizes asymptotically the control system (1)–(2) in a suitable neigbourhood of the optimal static point (s^*, x^*).

4. Numerical experiments. The numerical simulations are carried out in the computer algebra system Maple 7. Using the numerical values in Table 1, one gets

$$\begin{aligned} u^* &= 0.1046051865, & Q(u^*) &= Q_{\max} = 0.3066861492, \\ s^* &= 0.4293689739, & x^* &= 0.5235436753. \end{aligned}$$

Starting with

$$s(0) = 0.39, \quad x(0) = 0.49$$

the control system is solved numerically using the feedback (10) with $\delta = 0.8$. The left part of Figure 2 visualizes the trajectory in the (s, x) phase plane; the starting point is denoted by circle and N denotes the optimal static point on the invariant segment. The right part of Figure 2 shows the corresponding time profile of Q from (3); the horizontal line goes through the point $(u^*, Q(u^*))$.

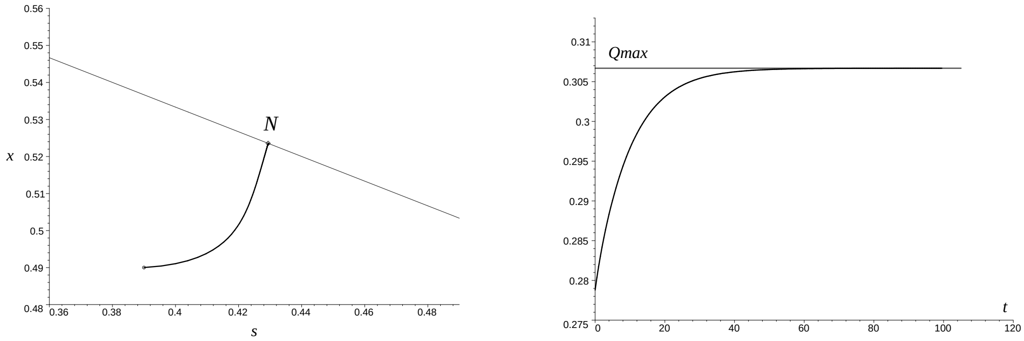

Figure 2: The trajectory in the (s, x) phase plane (left) and $Q(t)$ (right)

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АСИМПТОТИЧНА СТАБИЛИЗАЦИЯ НА БИОТЕХНОЛОГИЧЕН ПРОЦЕС СЪС СУБСТРАТНО ИНХИБИРАНЕ

Нели Димитрова, Михаил Кръстанов

Изследван е нелинеен динамичен модел на метанова ферментация, в който моделната функция на растеж на микроорганизмите отразява ефекта на субратно инхибиране. Предложена е непрекъсната обратна връзка, която стабилизира асимптотично процеса към оптимална точка, пресметната съгласно практически критерий. Представени са резултати от числови експерименти в системата за компютърна алгебра Maple.