МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2005
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2005
Proceedings of the Thirty Fourth Spring Conference of
the Union of Bulgarian Mathematicians
Borovets, April 6–9, 2005

INVARIANT DISTANCES AND METRICS IN COMPLEX ANALYSIS

Marek Jarnicki, Peter Pflug

The aim of this survey lecture is to present some research areas of the invariant distances and metrics theory. We discuss the main invariant objects and their relations to complex analysis.

1. Holomorphically contractible families of functions. An old method of complex analysis is to study properties of holomorphic mappings $F : G \longrightarrow D$ ($F \in \mathcal{O}(G, D)$) using appropriate pseudodistances. More precisely, we endow G and D with pseudodistances d_G and d_D, respectively, in such a way that every holomorphic mapping $F : G \longrightarrow D$ is a *contraction*, i.e.

(*) $$d_D(F(a), F(z)) \le d_G(a, z), \quad a, z \in G.$$

We expect that, if the pseudodistances d_G and d_D are reasonably chosen, then they describe some properties of $\mathcal{O}(G, D)$. This method, nowadays almost standard, has its roots in papers of Carathéodory from the twenties of the last century (cf. [3]). It has appeared that sometimes, instead of pseudodistances, one should consider more general objects, e.g. pluricomplex Green functions. In this way we are led to the following concept of *holomorphically contractible families of functions*:

Suppose that we are given a family $\underline{d} = (d_G)_G$ of functions $d_G : G \times G \longrightarrow \mathbb{R}_+$, where G runs over all (non-empty) domains in all $\mathbb{C}^n$'s (in fact, G can run over all connected complex manifolds or even over all connected complex analytic spaces). We say that $\underline{d}$ is *holomorphically contractible* if the following two conditions are satisfied:

(Normalization) for the unit disc $\mathbb{D} \subset \mathbb{C}$, the function $d_{\mathbb{D}}$ coincides with the *Möbius distance* $m_{\mathbb{D}}$, i.e.

$$d_{\mathbb{D}}(a, z) = m_{\mathbb{D}}(a, z) := \left|\frac{z - a}{1 - \overline{a}z}\right|, \quad a, z \in \mathbb{D};$$

(Contractibility) for any domains $G \subset \mathbb{C}^n$, $D \subset \mathbb{C}^m$, every mapping $F \in \mathcal{O}(G, D)$ satisfies (*).

In particular, if $F : G \longrightarrow D$ is biholomorphic, then $d_D(F(a), F(z)) = d_G(a, z)$, $a, z \in G$.

The main contractible families of functions are the following ones:

• *Möbius pseudodistance:*

$$c^*_G(a, z) := \sup\{m_{\mathbb{D}}(f(a), f(z)) : f \in \mathcal{O}(G, \mathbb{D})\} = \sup\{|f(z)| : f \in \mathcal{O}(G, \mathbb{D}),\ f(a) = 0\};$$

the function $c_G := \tanh^{-1} c_G^*$ is called the *Carathéodory pseudodistance*.

• *Higher order Möbius function:*

$$m_G^{(k)}(a,z) := \sup\{|f(z)|^{1/k} : f \in \mathcal{O}(G,\mathbb{D}),\ \mathrm{ord}_a f \geq k\} \quad (k \in \mathbb{N}),$$

where $\mathrm{ord}_a f$ denotes the order of zero of f at a.

• *Pluricomplex Green function:*

$$g_G(a,z) := \sup\{u(z) : \quad u : G \longrightarrow [0,1),\ \log u \in \mathcal{PSH}(G),\ \sup_{w\in G\setminus\{a\}} u(w)/\|w-a\| < +\infty\},$$

where $\mathcal{PSH}(G)$ denotes the family of all functions plurisubharmonic on G.

• *Lempert function:*

$$\begin{aligned}\widetilde{k}_G^*(a,z) &:= \inf\{m_{\mathbb{D}}(\lambda,\mu) : \quad \lambda,\mu \in \mathbb{D} : \exists_{\varphi\in\mathcal{O}(\mathbb{D},G)} : \varphi(\lambda) = a,\ \varphi(\mu) = z\}\\ &= \inf\{\mu \in [0,1) : \exists_{\varphi\in\mathcal{O}(\mathbb{D},G)} : \varphi(0) = a,\ \varphi(\mu) = z\}.\end{aligned}$$

It is well known that $c_G^* = m_G^{(1)} \leq m_G^{(k)} \leq g_G \leq \widetilde{k}_G^*$, and for any holomorphically contractible family $(d_G)_G$ we have $c_G^* \leq d_G \leq \widetilde{k}_G^*$, i.e. the Möbius family is minimal and the Lempert family is maximal. The pseudodistance

$$k_G := \sup\{d : \quad d : G \times G \longrightarrow \mathbb{R}_+ \text{ is a pseudodistance with } \tanh d \leq \widetilde{k}_G^*\}$$

is called the *Kobayashi pseudodistance*. Put $k_G^* := \tanh k_G$. The family $(k_G^*)_G$ is holomorphically contractible in the sense of our definition.

2. Holomorphically contractible families of pseudometrics. Parallel to the category of holomorphically contractible families of functions one studies holomorphically contractible families of pseudometrics. Suppose we are given a family $\underline{\delta} = (\delta_G)_G$ of *pseudometrics* $\delta_G : G \times \mathbb{C}^n \longrightarrow \mathbb{R}_+$,

$$\delta_G(a;\lambda X) = |\lambda|\delta_G(a;X), \quad (a,X) \in G \times \mathbb{C}^n,\ \lambda \in \mathbb{C},$$

where G runs over all domains $G \subset \mathbb{C}^n$. We say that $\underline{\delta}$ is *holomorphically contractible* if the following two conditions are satisfied:

(Normalization) $\delta_{\mathbb{D}}(a;X) = \gamma_{\mathbb{D}}(a;X) := \dfrac{|X|}{1-|a|^2}, \quad (a,X) \in \mathbb{D} \times \mathbb{C};$

(Contractibility) for any domains $G \subset \mathbb{C}^n$, $D \subset \mathbb{C}^m$ and every mapping $F \in \mathcal{O}(G,D)$, we have $\delta_D(F(a);F'(a)(X)) \leq \delta_G(a;X)$, $(a,X) \in G \times \mathbb{C}^n$.

In particular, if $F : G \longrightarrow D$ is biholomorphic, then $\delta_D(F(a);F'(a)(X)) = \delta_G(a;X)$, $(a,X) \in G \times \mathbb{C}^n$.

The following holomorphically contractible families of pseudometrics correspond to the above holomorphically contractible families of functions.

• *Carathéodory–Reiffen pseudometric*:

$$\gamma_G(a;X) := \sup\{|f'(a)(X)| : f \in \mathcal{O}(G,\mathbb{D}),\ f(a) = 0\}.$$

• *Higher order Reiffen pseudometric*:

$$\gamma_G^{(k)}(a;X) := \sup\left\{\left|\frac{1}{k!}f^{(k)}(a)(X)\right|^{1/k} : f \in \mathcal{O}(G,\mathbb{D}),\ \mathrm{ord}_a f \geq k\right\} \quad (k \in \mathbb{N}).$$

• *Azukawa pseudometric*:

$$A_G(a;X) := \limsup_{\mathbb{C}_*\ni\lambda\to 0} \frac{1}{|\lambda|} g_G(a, a+\lambda X).$$

• *Kobayashi–Royden pseudometric*:

$$\varkappa_G(a;X) := \inf\{\alpha \geq 0 : \exists_{\varphi\in\mathcal{O}(\mathbb{D},G)} : \varphi(0) = a, \ \alpha\varphi'(0) = X\}.$$

It is well known that $\gamma_G = \gamma_G^{(1)} \leq \gamma_G^{(k)} \leq A_G \leq \varkappa_G$, and for any holomorphically contractible family of pseudometrics $(\delta_G)_G$ we have $\gamma_G \leq \delta_G \leq \varkappa_G$, i.e. the Carathéodory–Reiffen pseudometric is minimal and the Kobayashi–Royden pseudometric is maximal.

3. Problems. Let us mention a few research areas which are important from the point of view of the theory.

• Effective formulas. The theory suffers from the lack of effective examples, i.e. examples of domains $G \subset \mathbb{C}^n$, for which all/some of basic contractible functions or pseudometrics can be effectively determined. In fact, even in the case $n = 1$, besides the unit disc and annuli, there are no examples. The only general class of domains for which effective formulas are known, are so-called *elementary Reinhardt domains* of the form

$$\{(z_1, \dots, z_n) : |z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n} < 1\}$$

where $\alpha_1, \dots, \alpha_n \in \mathbb{R}$.

• Hyperbolicity. Given a holomorphically contractible family of functions $\underline{d}$ (resp. pseudometrics $\underline{\delta}$) and a domain $G \subset \mathbb{C}^n$, find conditions under which $d_G(a, z) > 0$ for all $a, z \in G$, $a \neq z$ (resp. $\delta_G(a; X) > 0$ for all $(a, X) \in G \times \mathbb{C}^n$, $X \neq 0$). The question is essential only for unbounded domains because all bounded domains are hyperbolic with respect to all contractible families.

• Completeness. Given a holomorphically contractible family of distances $\underline{d}$ and a domain $G \subset \mathbb{C}^n$, find conditions under which the metric space (G, d_G) is (in certain sense) complete. Completeness is strictly connected with other holomorphic properties of G, e.g. with tautness.

• Boundary behavior and localization. Here we have a large variety of problems related to the asymptotic behavior of $d_G(a, z)$ when $z \longrightarrow b \in \partial G$, or $a, z \longrightarrow b \in \partial G$ (resp. of $\delta_G(a; X)$ when $a \longrightarrow b \in \partial G$). It has appeared that there are strong links between asymptotic behavior of certain contractible functions and boundary regularities of G (like strong pseudoconvexity or hyperconvexity).

• Geodesics. Given a holomorphically contractible family of functions $\underline{d}$ (resp. pseudometrics $\underline{\delta}$), a domain $G \subset \mathbb{C}^n$, and two points $a, b \in G$ (resp. a point $a \in G$ and a vector $X \in \mathbb{C}^n$), decide whether there exist a holomorphic mapping $\varphi : \mathbb{D} \longrightarrow G$ and points $\lambda, \mu \in \mathbb{D}$ such that $\varphi(\lambda) = a$, $\varphi(\mu) = b$, and $d_G(a, b) = m_{\mathbb{D}}(\lambda, \mu)$ (resp. a holomorphic mapping $\varphi : \mathbb{D} \longrightarrow G$, a point $\lambda \in \mathbb{D}$, and a number $\alpha \in \mathbb{C}$ such that $\varphi(\lambda) = a$, $\alpha\varphi'(\lambda) = X$, and $\delta_G(a; X) = \gamma_{\mathbb{D}}(\lambda; \alpha)$. The answer is always positive in the class of convex domains. Many other cases are still open.

• Product property. Given a holomorphically contractible family of functions $\underline{d}$ (resp. pseudometrics $\underline{\delta}$), we ask whether for all domains $G \subset \mathbb{C}^n$, $D \subset \mathbb{C}^m$ we have

$$d_{G\times D}((a, b), (z, w)) = \max\{d_G(a, z), \ d_D(b, w)\}, \quad (a, b), (z, w) \in G \times D$$

(resp.

$$\delta_{G\times D}((a, b); (X, Y)) = \max\{\delta_G(a; X), \ \delta_D(b; Y)\}, \quad (a, b) \in G \times D, \ (X, Y) \in \mathbb{C}^n \times \mathbb{C}^m).$$

It has been proved that the families $\underline{c}^*$, $\underline{g}$, $\underline{k}^*$, $\underline{\gamma}$, $\underline{A}$, $\underline{\varkappa}$ have the product property. It is also known that for $k \geq 2$ the families $\underline{m}^{(k)}$, $\underline{\gamma}^{(k)}$ have no product property.

• Different contractibility conditions. Our contractibility condition deals with all holomorphic mappings $F : G \longrightarrow D$. One can consider weaker conditions which require the contractibility only for some subclasses of holomorphic mappings, e.g. injective holomorphic mappings or even only biholomorphic mappings. The latter case allows, for instance, to consider Bergman pseudodistance and pseudometric as contractible objects.

• Lempert theorem. Let $\mathfrak{L}_n$ be the class of all domains $G \subset \mathbb{C}^n$ for which $c^*_G \equiv \widetilde{k}^*_G$ and $\gamma_G \equiv \varkappa_G$. In particular, if $G \in \mathfrak{L}_n$, then all holomorphically contractible functions (reps. pseudometrics) coincide on G. It is clear that $\mathfrak{L}_n$ is invariant under biholomorphic mappings. Moreover, if a domain $G \subset \mathbb{C}^n$ may be exhausted by domains from $\mathfrak{L}_n$, then $G \in \mathfrak{L}_n$.

The fundamental Lempert theorem ([10, 11]) says that *all convex domains belong to* $\mathfrak{L}_n$. For more than 20 years the following conjecture was open. *Any bounded pseudoconvex domain $G \in \mathfrak{L}_n$ may be exhausted by domains biholomorphic to convex domains.* The first counterexample was recently constructed in a series of papers by J. Agler, C. Costara, and N. J. Young [1–6] This is the so-called *symmetrized bidisc*

$$\mathbb{G}_2 := \{(\lambda_1 + \lambda_2, \lambda_1\lambda_2) : \lambda_1, \lambda_2 \in \mathbb{D}\}.$$

One should mention that for $n \geq 3$ the holomorphic geometry of the *symmetrized n–disc*

$$\mathbb{G}_n := \{(\sigma_1(\lambda), \dots, \sigma_n(\lambda)) : \lambda \in \mathbb{D}^n\},$$

where $\sigma_1, \dots, \sigma_n$ are standard fundamental symmetric polynomials, remains still unclear.

4. Generalized holomorphically contractible families. The Möbius and Lempert functions are symmetric. The higher Möbius functions and the Green function are not symmetric in general. Their definitions distinguish one point (pole) at which we impose growth conditions. From that point of view it is natural to investigate objects with more general growth conditions. For instance, the Green function g_G may be generalized as follows. Let $G \subset \mathbb{C}^n$ be a domain and let $\boldsymbol{p} : G \longrightarrow \mathbb{R}_+$ be an arbitrary function. Put $|\boldsymbol{p}| := \{z \in G : \boldsymbol{p}(z) > 0\}$ and define

$$g_G(\boldsymbol{p}, z) := \sup\{u(z) : \ u : G \longrightarrow [0, 1),\ \log u \in \mathcal{PSH}(G), \\ \forall_{a\in|\boldsymbol{p}|} : \sup_{w\in G\setminus\{a\}} u(w)/\|w - a\|^{\boldsymbol{p}(a)} < +\infty\}, \quad z \in G.$$

The function $g_G(\boldsymbol{p}, \cdot)$ is called the *generalized pluricomplex Green function with poles (weights, pole function)* $\boldsymbol{p}$.

In the case where $\boldsymbol{p} = \chi_A$ = the characteristic function of a set $A \subset G$, we put $g_G(A, \cdot) := g_G(\chi_A, \cdot)$. Obviously, $g_G(\{a\}, \cdot) = g_G(a, \cdot)$. In the case where the set $|\boldsymbol{p}|$ is finite, the function $g_G(\boldsymbol{p}, \cdot)$ was first introduced by P. Lelong in [9]. The generalized pluricomplex Green function was recently intensively studied by many authors.

Using similar ideas, one can generalize the Möbius function. Let $G \subset \mathbb{C}^n$ be a domain and let $\boldsymbol{p} : G \longrightarrow \mathbb{Z}_+$ be an arbitrary function. Define

$$m_G(\boldsymbol{p}, z) := \sup\{|f(z)| : f \in \mathcal{O}(G, \mathbb{D}),\ \forall_{a\in|\boldsymbol{p}|} : \operatorname{ord}_a f \geq \boldsymbol{p}(a)\}, \quad z \in G.$$

The function $m_G(\boldsymbol{p}, \cdot)$ is called the *generalized Möbius function with weights* $\boldsymbol{p}$. Similarly as in the case of the generalized Green function, we put $m_G(A, \cdot) := m_G(\chi_A, \cdot)$ and $m_G(a, \cdot) := m_G(\{a\}, \cdot)$. Obviously, $m_G(a, \cdot) = c^*_G(a, \cdot)$, $a \in G$. More generally, $m_G(k\chi_{\{a\}}, \cdot) = [m^{(k)}_G(a, \cdot)]^k$. It is clear that $m_G(\boldsymbol{p}, \cdot) \leq g_G(\boldsymbol{p}, \cdot)$.

The above two generalizations lead us to the following definition. A family $\underline{d} = (d_G)_G$ of functions $d_G : \mathbb{R}^G_+ \times G \longrightarrow \mathbb{R}_+$ is said to be a *generalized holomorphically contractible* *family* if the following three conditions are satisfied:

(Normalization) $\prod_{a\in\mathbb{D}}[m_{\mathbb{D}}(a,z)]^{\boldsymbol{p}(a)} \le d_{\mathbb{D}}(\boldsymbol{p},z) \le \inf_{a\in\mathbb{D}}[m_{\mathbb{D}}(a,z)]^{\boldsymbol{p}(a)}$, $(\boldsymbol{p},z) \in \mathbb{R}_+^{\mathbb{D}} \times \mathbb{D}$;

(Contractibility) for any domains $G \subset \mathbb{C}^n$, $D \subset \mathbb{C}^m$, $F \in \mathcal{O}(G,D)$, and $\boldsymbol{q} : D \longrightarrow \mathbb{R}_+$, we have $d_D(\boldsymbol{q}, F(z)) \le d_G(\boldsymbol{q} \circ F, z)$, $z \in G$;

(Monotonicity) for any domain D and $\boldsymbol{p}, \boldsymbol{q} : G \longrightarrow \mathbb{R}_+$, if $\boldsymbol{p} \le \boldsymbol{q}$, then $d_G(\boldsymbol{q}, \cdot) \le d_G(\boldsymbol{p}, \cdot)$.

If in the above definition one considers only integer-valued weights (like in the case of the generalized Möbius function), then we get the definition of a *generalized holomorphically contractible family with integer-valued weights*. As usually, we put $d_G(A,\cdot) := d_G(\chi_A,\cdot)$ and $d_G(a,\cdot) := d_G(\{a\},\cdot)$.

One can prove that the generalized Green and Möbius functions are generalized holomorphically contractible families in the sense of the above definition. Moreover, one can prove that there exist minimal and maximal generalized holomorphically contractible families.

More details related to the above sketched theory may be found in our monograph [7] (*Invariant Distances and Metrics in Complex Analysis*, de Gruyter Expositions in Mathematics 9, Walter de Gruyter 1993) and recent survey article [8] (*Invariant Distances and Metrics in Complex Analysis — revisited*, Diss. Math. 430 (2005), 1–192).

REFERENCES

[1] J. Agler, N. J. Young. A Schwarz lemma for the symmetrized bidisc. *Bull. London Math. Soc.*, **33** (2001), 175-186.

[2] J. Agler, N. J. Young. The hyperbolic geometry of the symmetrized bidisc. *J. Geom. Anal.*, **14** (2004).

[3] C. Carathéodory. Über eine spezielle Metrik, die in der Theorie der analytischen Funktionen auftritt. *Atti Pontifica Acad. Sc., Nuovi Lincei*, **80** (1927), 135–141.

[4] C. Costara. Dissertation, Private communication, 2003.

[5] C. Costara. The symmetrized bidisc and Lempert's theorem. *Bull. London Math. Soc.*, **36** (2004), 656–662.

[6] C. Costara. On the 2×2 spectral Nevanlinna–Pick problem. Preprint, 2004.

[7] M. Jarnicki, P. Pflug. Invariant Distances and Metrics in Complex Analysis. de Gruyter Expositions in Mathematics 9, Walter de Gruyter 1993.

[8] M. Jarnicki, P. Pflug. Invariant Distances and Metrics in Complex Analysis — revisited. *Diss. Math.*, **430** (2005), 1–192.

[9] P. Lelong. Fonction de Green pluricomplexe et lemme de Schwarz dans les espaces de Banach. *J. Math. Pures Appl.*, **68** (1989), 319–347.

[10] L. Lempert. La métrique de Kobayashi et la représentation des domaines sur la boule. *Bull. Soc. Math. France*, **109** (1981), 427–474.

[11] L. Lempert. Holomorphic retracts and intrinsic metrics in convex domains. *Analysis Mathematica*, **8** (1982), 257–261.

Marek Jarnicki
Jagiellonian University
Institute of Mathematics
Reymonta 4, 30-059 Kraków, Poland
e-mail: Marek.Jarnicki@im.uj.edu.pl

Peter Pflug
Carl von Ossietzky Universität Oldenburg
Institut für Mathematik
Postfach 2503, D-26111 Oldenburg, Germany
e-mail: pflug@mathematik.uni-oldenburg.de

ИНВАРИАНТНИ РАЗСТОЯНИЯ И МЕТРИКИ В КОМПЛЕКСНИЯ АНАЛИЗ

Марек Ярницки, Петер Пфлуг

Тази обзорна лекция е посветена на някои направления в теорията на инвариантните разстояния и метрики. В нея се дискутират основните инвариантни обекти и техните връзки с комплексния анализ.