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NEW TERNARY AND QUATERNARY EQUIDISTANT CONSTANT WEIGHT CODES*

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We consider the problem of finding bounds on the size of ternary and quaternary equidistant constant weight codes. We present a computer realization of an algorithm for solving the maximum clique problem. We use it for finding the exact values of the maximum size for ternary and quaternary equidistant constant weight codes for all open cases for $n \leq 10$ are found.

1. Introduction. Let Z_q denote the set $\{0, 1, \ldots, q-1\}$ and let Z_q^n be the set of all n-tuples over Z_q and $Z_q^{n,w}$ be the set of n-tuples over Z_q of Hamming weight w.

A code is called *equidistant* if all the distances between distinct codewords are d. A code is called *constant weight* if all the codewords have the same weight w. An $(n, M, d)_q$ equidistant code is a code over Z_q of length n, cardinality M and distance d.

Let $B_q(n, d)$ denote the maximum number of codewords in an equidistant code over Z_q of length n and distance d (called an $(n, M, d)_q$ equidistant code) and $B_q(n, d, w)$ denote the maximum number of codewords in an equidistant constant weight code over Z_q of length n, distance d, and weight w (called an $(n, M, d, w)_q$ equidistant constant weight code, ECWC). The study of these functions is a maximum clique problem.

Equidistant codes have been investigated in [5, 6, 8, 9, 11]. A few papers study codes which are both equidistant and of constant weight. Some works published on this topic are [4, 7, 12, 2, 1, 3] Tables of the best known bounds for ternary and quaternary equidistant constant weight codes with parameters $2 \leq w < n$ and $3 \leq n \leq 10$ is presented in [2, 3].

In this paper we consider bounds on $B_q(n, d, w)$ for $q = 3, 4$. Upper and lower bounds for ECWC are given in Section 2. The constructions of equidistant codes by computer search and results on $B_q(n, d, w)$ for $q = 3, 4$ are given in section 3. We find the exact values of $B_3(n, d, w)$ and $B_4(n, d, w)$ for all open cases for $n \leq 10$ and $2 \leq w < n$ [2, 3].

2. General Bounds. The equidistant codes and ECWC are closely related as it is shown by following theorem:

Theorem 1. [4] *It is true that* $B_q(n, d) = 1 + B_q(n, d, d)$.

Some general bounds for equidistant codes and ECWC are given by the following theorems:

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Theorem 2. (Delsarte) $B_q(n,d) \leq (q-1)n+1$.

Theorem 3. *It is true that* $B_q(n,n)=q$, $B_q(n,n,w) \leq q$, $B_q(n,d,n)=B_{q-1}(n,d)$ $B_q(n+1,d,w) \geq B_q(n,d,w)$ *and* $B_q(n+1,d,w+1) \geq B_q(n,d,w)$.

Theorem 4. (the Johnson bounds for ECWC) *The maximum number of codewords in a q-ary ECWC satisfy the inequalities:*

$$B_q(n,d,w) \leq \frac{n}{n-w} B_q(n-1,d,w) \ \text{ and } B_q(n,d,w) \leq \frac{n(q-1)}{w} B_q(n-1,d,w-1).$$

Theorem 5. [4] *For* $k=1,2,\ldots,n$, *if* $P_k^2(w) > P_k(d)P_k(0)$, *then*

$$B_q(n,d,w) \leq \frac{P_k^2(0)-P_k(d)P_k(0)}{P_k^2(w)-P_k(d)P_k(0)}.$$

Here $P_k(x)$ is the Krawtchouk polynomial defined by

$$P_k(x)=\sum_{i=0}^{k}\binom{x}{i}\binom{n-x}{k-i}(-1)^i(q-1)^{k-i} \quad \text{and} \quad P_k(0)=\binom{n}{k}(q-1)^k.$$

Theorem 6. [7] $B_q(q+1,q,q-1) \leq (q^2+q)/2$.

There exist the following families of special equidistant codes for $q=3,4$ and $d=3,4$.

Theorem 7. [1] $B_3(n,3)=9$ *precisely when* $3 \leq q \leq 9$. $B_q(n,3)=q$ *precisely when* $q>9$. *Codes with such parameters are unique (up to equivalence).*

Theorem 8. [1] $B_4(n,4)=16$ *precisely when* $4 \leq q \leq 16$ *and* $4 \leq n \leq 33$. $B_q(n,4)$ $=q$ *precisely when* $q>16$. *Codes with such parameters are unique (up to equivalence).*

3. Methods and results of finding ECWC. A simple graph $G=(V;E)$ is a set of vertices V and set of unordered pairs of distinct elements of V called edges. Not all graphs are simple. Sometimes a pair of vertices are connected by multiple edge yielding a multigraph. Vertices connected to themselves by a edge called a loop, yeilding a pseudograph. Finally, edges can also be given a direction yielding a directed graph (or digraph). A graph having a weight, or number, associated with each edge is called weighted graph.

A clique of graph is a set of vertices, any two of which are adjacent. Maximal clique is a clique which vertices is not a subset of the vertices of a larger clique. Maximum clique is the largest clique in the graph. Maximum-weight clique is a clique with maximum weight.

Maximum-weigh clique problem:

– the vertices have weights, and one wants to find a clique with maximum weight;

– it is NP-hard;

– the clique is not necessarily a maximum clique of the underlying unweighted graph, but it is certainly maximal.

A fast algorithm for solving this problem is given in [10], [13].

In this work we present a variant of this algorithm for ECWC.

We assume some order for the vertices $V=v_1,v_2,\ldots,v_n$. S_n that contain v_n , then cliques in S_{n-1} that contain v_{n-1}. Let $S_1=\{v_1,v_2,\ldots,v_i\} \subseteq V$. Considering cliques in V, we define the function $c(i)$ to be the size of the maximum clique in the subgraph induced by S_1. Obviously, for every $i = 1, \ldots, n-1$ we have either $c(i+1) = c(i)$ or $c(i+1) = c(i) + 1$. Moreover, $c(i+1) = c(i) + 1$ if there exists a clique in S_{i+1} of size $c(i) + 1$ that includes vertex v_{i+1}.

Then we calculate the values of $c(i)$ starting from $c(1) = 1$ up, and stores the values found. Then the algorithm is searching for a clique of size $c(i) + 1$ within S_{i+1}, it has formed a clique W and is considering adding vertex v_j, it can prune the search if $\mid W \mid + c(j) \leq c(i)$. As j is chosen to be the largest index in the set of vertices to be considered, it follows that a clique of size $c(i) + 1$ that contains W cannot exist in S_{i+1}. Trivially, if it finds a clique of size $c(i) + 1$, it can prune the whole search and start calculating $c(i+2)$. Table $c[i]$ gives the largest clique includes the vertex v_i. When we are searching for all maximum cliques, we first determine the size of the maximum cliques, and then starts the search again at the suitable position.

Let C be an $(n, M, d, w)_q$ ECWC. Our approach is based on the observation that an (n, M, d, w) code C can be shortened to an $(n-1, M, d, w)_q$ code C_0. Conversely, if we want to construct an $(n, M, d, w)_q$ code C, we only need to consider lengthening of the $(n-1, M, d, w)_q$ code C_0. The main problems we decide in this paper is code construction of some $(n, M, d, w)_q$ ECWC.

This is a maximal clique problem and we use a computer realization of the described algorithm. The search space will only be the vectors which are at a distance exactly d from the code C_0 and have exactly weight w. We will only have to care about the distance between codewords and for their weights. In the ternary (quaternary) case we can construct the graph whose vertices represent ternary (quaternary) vectors of length n. We join two vertices by an edge if and only if the Hamming distance between the vectors is exactly d and their weight is exactly w. Then what we are interested in is the quantity $B_q(n, d, w)$, the size of the largest clique in this graph.

The following theorem is derived from Theorem 4:

Theorem 9. *Any (n, M, d, w) ECWC code C contains $(n-1, M', d, w)$ codes with $M' \geq \lceil M \frac{n-w}{n} \rceil$ codewords.*

The results for ternary and quaternary ECWC are obtained by a computer program based on the method in [13]. We made our own realization for ECWC. The upper bounds for ECWC which we use for our computer research are obtained from theorems presented in Section 2. The exact values for ECWC over an alphabet of three and four elements of length $n \leq 10$ are displayed in Table 1 and Table 2. New results in the tables are denoted by "$\star$". All open cases for ternary and quaternary ECWC of length $n \leq 10$ from [2, 3] are solved.

Table 1. Bounds on $B_3\,(n, d, w)$ for $4 < w < 9$ and $n = 10$

n	w	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$	$d=9$	$d=10$
10	5	8	8	7	$\star$**12**	$\star$**8**	4	2	2
	6	8	8	7	$\star$**14**	$\star$**8**	5	3	2
	7	8	8	8	$\star$**12**	$\star$**9**	5	3	2
	8	8	8	8	$\star$**15**	$\star$**10**	5	2	2

Table 2. Bounds on $B_4(n, d, w)$ for $8 \geq n \leq 10$.

n	w	$d = 3$	$d = 4$	$d = 5$	$d = 6$	$d = 7$	$d = 8$	$d = 9$	$d = 10$
8	5	9	15	11	$\star$**10**	5	2		
	6	9	15	9	$\star$**12**	5	4		
	7	9	15	11	$\star$**12**	4	3		
9	4	9	15	9	$\star$**9**	3	2		
	5	9	15	$\star$**11**	$\star$**10**	9	3	2	
	6	9	15	$\star$**11**	$\star$**12**	$\star$**11**	5	3	
	7	9	15	$\star$**11**	$\star$**12**	$\star$**11**	5	3	
	8	9	15	$\star$**11**	$\star$**12**	$\star$**11**	4	3	
10	5	9	15	$\star$**11**	$\star$**12**	$\star$**10**	6	2	2
	6	9	15	$\star$**11**	$\star$**14**	$\star$**11**	10	5	2
	7	9	15	$\star$**11**	$\star$**12**	$\star$**11**	$\star$**14**	5	3
	8	9	15	$\star$**11**	$\star$**15**	$\star$**11**	15	5	3
	9	9	15	$\star$**11**	$\star$**12**	$\star$**11**	8	4	3

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НОВИ ЕКВИДИСТАНТНИ КОНСТАНТНО-ТЕГЛОВНИ КОДОВЕ НАД АЗБУКА С ТРИ И ЧЕТИРИ ЕЛЕМЕНТА

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Разглеждаме проблема за намиране на граници за обема на еквидистантни константно-тегловни кодове над азбука с три и четири елемента. За решаването на този проблем се използва алгоритъм за намиране на максимално клика в граф. Намерени са точни стойности за максималния обем на еквидистантни константно-тегловни кодове над азбука с три и четири елемента за всички отворени случаи при $n \leq 10$.