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SOME INTEGRAL GEOMETRIC RESULTS ON SETS OF CIRCLES IN THE SIMPLY ISOTROPIC SPACE*

Adrijan V. Borisov, Margarita G. Spirova

The measurability with respect to the group of the general simply isotropic similitudes of sets of circles (elliptic and parabolic) is studied and also some formulas for the density are given.

1. Introduction. The simply isotropic space ${I_3}^{(1)}$ is defined as a projective space $\mathbb{P}_3(\mathbb{R})$ with an absolute plane ω and two complex conjugate straight lines f_1, f_2 in ω with a (real) intersection point F. All regular projectivities transforming the absolute figure into itself form the 8-parametric group G_8 of the general simply isotropic similitudes. Passing on to affine coordinates (x, y, z) any similitude of G_8 can be written in the form

$$\begin{aligned} \overline{x} &= c_1 + c_7(x\cos\varphi - y\sin\varphi), \\ \overline{y} &= c_2 + c_7(x\sin\varphi + y\cos\varphi), \\ \overline{z} &= c_3 + c_4x + c_5y + c_6z, \end{aligned} \tag{1}$$

where $c_1, c_2, c_3, c_4, c_5, c_6 \neq 0, c_7 > 0$ and φ are real parameters. We emphasize that more of the common material of the geometry of $I_3^{(1)}$ can be found in [3].

Using some basic concepts of the integral geometry in the sense of M. I. Stoka [4], G. I. Drinfel'd [2], we study the measurability of sets of circles in ${I_3}^{(1)}$ with respect to G_8.

2. Measurability of a set of circles of elliptic type. The conic k of one part in the nonisotropic plane ε is called a *circle of elliptic type* if the infinite points of k coincide with the intersection points of ε with f_1 and f_2 [3; p. 70]. We note the fact that any nonisotropic plane ε, which is not a tangential plane of a sphere of parabolic type Σ and $\varepsilon \cap \Sigma \neq \emptyset$, intersects Σ in a circle k of elliptic type [3; p. 72] (Fig. 1.a)).

Now, let k be a circle of elliptic type defined by the equations

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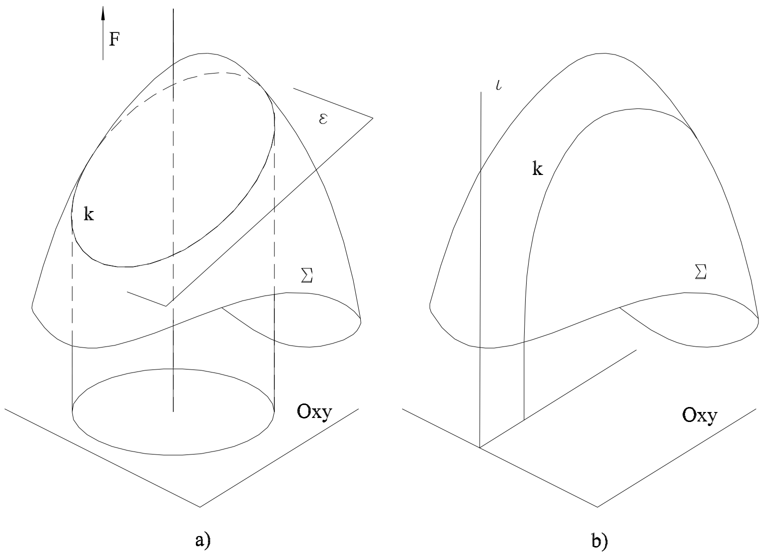

Fig. 1

$$\text{(2)} \qquad k: \begin{array}{l} z = R(x^2+y^2) + \alpha x + \beta y + \gamma, \\ z = ux + vy + w, \end{array}$$

where $R \neq 0$ and $(\alpha-u)^2 + (\beta-v)^2 - 4R(\gamma-w) > 0$. Under the action of (1) the circle $k(R,\alpha,\beta,\gamma,u,v,w)$ is transformed into the circle $\overline{k}(\overline{R},\overline{\alpha},\overline{\beta},\overline{\gamma},\overline{u},\overline{v},\overline{w})$ according to

$$\begin{aligned}
&\overline{R} = c_6 c_7^{-1} R,\\
&\overline{\alpha} = c_7^{-1}[-2c_1c_6c_7^{-1}R + (c_6\alpha + c_4)\cos\varphi - (c_6\beta + c_5)\sin\varphi],\\
&\overline{\beta} = c_7^{-1}[-2c_2c_6c_7^{-1}R + (c_6\alpha + c_4)\sin\varphi + (c_6\beta + c_5)\cos\varphi],\\
&\overline{\gamma} = c_7^{-1}[(c_1^2 + c_2^2)c_6c_7^{-1}R + c_3c_7 + c_6c_7\gamma - (c_6\alpha + c_4)(c_1\cos\varphi + c_2\sin\varphi) +\\
\text{(3)} \qquad &\qquad + (c_6\beta + c_5)(c_1\sin\varphi - c_2\cos\varphi)],\\
&\overline{u} = c_7^{-1}[(c_6u + c_4)\cos\varphi - (c_6v + c_5)\sin\varphi],\\
&\overline{v} = c_7^{-1}[(c_6u + c_4)\sin\varphi + (c_6v + c_5)\cos\varphi],\\
&\overline{w} = c_7^{-1}[c_3c_7 + c_6c_7w - (c_6u + c_4)(c_1\cos\varphi + c_2\sin\varphi) +\\
&\qquad + (c_6v + c_5)(c_1\sin\varphi - c_2\cos\varphi)].
\end{aligned}$$

The transformations (3) form the so-called associated group $\overline{G_8}$ of G_8 [4; p. 34]. $\overline{G_8}$ is isomorphic to G_8 and the invariant density under G_8 of the circles of elliptic type k, if it exists, coincides with the invariant density under $\overline{G_8}$ of the points $(R, \alpha, \beta, \gamma, u, v, w)$ in the set of parameters [4; p. 33]. The associated group $\overline{G_8}$ has the infinitesimal operators

$$\begin{aligned}
Y_1 &= 2R\frac{\partial}{\partial\alpha} + \alpha\frac{\partial}{\partial\gamma} + u\frac{\partial}{\partial w},\ Y_2 = 2R\frac{\partial}{\partial\beta} + \beta\frac{\partial}{\partial\gamma} + v\frac{\partial}{\partial w},\\
Y_3 &= \frac{\partial}{\partial\gamma} + \frac{\partial}{\partial w},\ Y_4 = \frac{\partial}{\partial\alpha} + \frac{\partial}{\partial u},\ Y_5 = \frac{\partial}{\partial\beta} + \frac{\partial}{\partial v},\\
Y_6 &= R\frac{\partial}{\partial R} + \alpha\frac{\partial}{\partial\alpha} + \beta\frac{\partial}{\partial\beta} + \gamma\frac{\partial}{\partial\gamma} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w},\\
Y_7 &= 2R\frac{\partial}{\partial R} + \alpha\frac{\partial}{\partial\alpha} + \beta\frac{\partial}{\partial\beta} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v},\\
Y_8 &= -\beta\frac{\partial}{\partial\alpha} + \alpha\frac{\partial}{\partial\beta} - v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}.
\end{aligned}$$

The integral invariant function $f = f(R, \alpha, \beta, \gamma, u, v, w)$ satisfies the system of R. Deltheil [1; p. 28]

$$Y_1(f) = 0,\ Y_2(f) = 0,\ Y_3(f) = 0,\ Y_4(f) = 0,\ Y_5(f) = 0,$$

$$Y_6(f) + 7f = 0,\ Y_7(f) + 4f = 0,\ Y_8(f) = 0$$

and it has the form

$$f = \frac{cR}{[(\alpha-u)^2 + (\beta-v)^2 - 4R(\gamma-w)]^4},$$

where $c = const \neq 0$.

Thus, we can state:

Theorem 1. *A set of circles of elliptic type* (2) *is measurable with respect to the group* G_8 *and has the invariant density*

$$dk = \frac{|R| dR \wedge d\alpha \wedge d\beta \wedge d\gamma \wedge du \wedge dv \wedge dw}{[(\alpha-u)^2 + (\beta-v)^2 - 4R(\gamma-w)]^4}. \tag{4}$$

Remark 1. The orthogonal projection of the circle of elliptic type k on the coordinate plane Oxy is the Euclidean circle (Fig. 1.a)).

$$\widetilde{k}:\ R(x^2 + y^2) + (\alpha-u)x + (\beta-v)y + \gamma - w = 0, z = 0 \tag{5}$$

with the center

$$O\left(-\frac{\alpha-u}{2R}, -\frac{\beta-v}{2R}, 0\right)$$

and the radius

$$r = \frac{1}{2|R|}\sqrt{(\alpha-u)^2 + (\beta-v)^2 - 4R(\gamma-w)}. \tag{6}$$

The quantity r is called the *radius of the circle of elliptic type* k [3; p. 70].

By (4) and (6) we have

$$dk = \frac{1}{256r^8|R|^7}\, dR \wedge d\alpha \wedge d\beta \wedge d\gamma \wedge du \wedge dv \wedge dw. \tag{7}$$

Remark 2. The coordinate plane Oxy is an Euclidean plane and into Oxy the circles $\widetilde{k}$ defined by (5) have under the group $\widetilde{G_4}$ of the similitudes

$$\begin{aligned}\overline{x} &= c_1 + c_7(x\cos\varphi - y\sin\varphi)\\ \overline{y} &= c_2 + c_7(x\sin\varphi + y\cos\varphi)\end{aligned}$$

the invariant density [4; p. 167]

$$d\widetilde{k} = \frac{dx_0 \wedge dy_0 \wedge d\gamma_0}{r^4}, \tag{8}$$

where

$$x_0 = -\frac{\alpha - u}{2R},\ y_0 = -\frac{\beta - v}{2R},\ \gamma_0 = \frac{\gamma - w}{R}. \tag{9}$$

Differentiating (9), we have

$$\begin{aligned}\frac{1}{2R}d\alpha &= -dx_0 + \frac{\alpha - u}{2R^2}\, dR + \frac{1}{2R}\, du,\\ \frac{1}{2R}d\beta &= -dy_0 + \frac{\beta - v}{2R^2}\, dR + \frac{1}{2R}\, dv,\\ \frac{1}{R}d\gamma &= d\gamma_0 + \frac{\gamma - w}{R^2}\, dR + \frac{1}{R}\, dw.\end{aligned} \tag{10}$$

By exterior product of (10) and $dR \wedge du \wedge dv \wedge dw$ we obtain

$$dR \wedge d\alpha \wedge d\beta \wedge d\gamma \wedge du \wedge dv \wedge dw = 4R^3 dR \wedge du \wedge dv \wedge dw \wedge dx_0 \wedge dy_0 \wedge d\gamma_0. \tag{11}$$

From (7), (8) and (11) it follows that

$$dk = \frac{1}{64r^4R^4}\, dR \wedge du \wedge dv \wedge dw \wedge d\widetilde{k}. \tag{12}$$

Remark 3. Denote by ψ the angle between both intersecting nonisotropic planes ε and Oxy. Then we have [3; p. 17]

$$\psi = +\sqrt{u^2 + v^2}$$

and, therefore,

$$\psi d\psi = udu + vdv.$$

From here it follows that

$$du \wedge dv = \frac{\psi}{u}\, d\psi \wedge dv = \frac{\psi}{v} du \wedge d\psi$$

and the formula of the density (12) becomes

$$\begin{aligned}dk &= \frac{\psi}{64r^4R^4|u|}\, dR \wedge d\psi \wedge dv \wedge dw \wedge \widetilde{k},\\ dk &= \frac{\psi}{64r^4R^4|v|}\, dR \wedge du \wedge d\psi \wedge dw \wedge \widetilde{k},\end{aligned} \tag{13}$$

respectively.

We summarize the foregoing results in the following

Theorem 2. *The density for the circles of the elliptic type* (4) *satisfies the relations* (7)*,* (12) *and* (13)*.*

3. Measurability of sets of circles of parabolic type. The conic k in the isotropic plane ι is called a *circle of the parabolic type* if k touches the absolute plane ω at the absolute point F [3; p. 70]. Any isotropic plane ι intersects a sphere of parabolic type Σ in a circle k of parabolic type [3; p. 72] (Fig. 1.b)).

Let k be a circle of parabolic type defined by the equations

$$k: \quad \begin{array}{c} z = R(x^2+y^2) + \alpha x + \beta y + \gamma, \\ ux + vy + 1 = 0, \end{array}$$

where $R \neq 0$, and $(u, v) \neq (0, 0)$. Now, the corresponding associated group $\overline{G_8}$ has the infinitesimal operators

$$Y_1 = -2R\frac{\partial}{\partial\alpha} - \alpha\frac{\partial}{\partial\gamma} + u^2\frac{\partial}{\partial u} + uv\frac{\partial}{\partial v},\ Y_2 = -2R\frac{\partial}{\partial\beta} - \beta\frac{\partial}{\partial\gamma} + uv\frac{\partial}{\partial u} + v^2\frac{\partial}{\partial v},$$

$$Y_3 = \frac{\partial}{\partial\gamma},\ Y_4 = \frac{\partial}{\partial\alpha},\ Y_5 = \frac{\partial}{\partial\beta},\ Y_6 = R\frac{\partial}{\partial R} + \alpha\frac{\partial}{\partial\alpha} + \beta\frac{\partial}{\partial\beta} + \gamma\frac{\partial}{\partial\gamma},$$

$$Y_7 = 2R\frac{\partial}{\partial R} + \alpha\frac{\partial}{\partial\alpha} + \beta\frac{\partial}{\partial\beta} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v},\ Y_8 = -\beta\frac{\partial}{\partial\alpha} + \alpha\frac{\partial}{\partial\beta} - v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}.$$

It is easy to verify that the system of R. Deltheil

$$Y_1(f) + 3uf = 0,\ Y_2(f) + 3vf = 0,\ Y_3(f) = 0,\ Y_4(f) = 0,$$

$$Y_5(f) = 0,\ Y_6(f) + 4f = 0,\ Y_7(f) + 6f = 0,\ Y_8(f) = 0$$

has the solution

$$f(R, \alpha, \beta, \gamma, u, v) = 0.$$

From here it follows

Theorem 3. *Sets of circles of parabolic type are not measurable with respect to the group* G_8*.*

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Margarita Georgieva Spirova
Dept. of Descriptive Geometry
Univ. of Architecture, Civil Eng. and Geodesy
1, Christo Smirnenski Blvd.
1046 Sofia, Bulgaria
e-mail: spirov_fgs@uacg.bg

Adrijan Varbanov Borisov
Dept. of Mathematics
Neofit Rilski South-West University
66, Ivan Mihailov Str.
2700 Blagoevgrad, Bulgaria
e-mail: adribor@aix.swu.bg

ИНТЕГРАЛНО-ГЕОМЕТРИЧНИ РЕЗУЛТАТИ ЗА МНОЖЕСТВА ОТ СФЕРИ В ПРОСТО ИЗОТРОПНО ПРОСРТАНСТВО

Адриян В. Борисов, Маргарита Г. Спирова

Изследвана е измеримостта на множества сфери (елиптични и параболични) относно групата на общите просто-изотропни подобности и са получени формули за съответната гъстота.