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ON VERY STRONG PSEUDOCONVEXITY AND VERY STRONG PSEUDOMONOTONICITY

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In this paper we introduce new concepts of a very strongly pseudoconvex function and a very strongly pseudomonotone map. It is shown that a differentiable function is very strongly pseudoconvex if and only if its gradient map is very strongly pseudomonotone. Characterizations of the very strongly pseudoconvex functions and the very strongly pseudomonotone maps are obtained.

1. Introduction It is well-known that a differentiable function f is convex if and only if its gradient map ∇f is monotone. This nice property was carried over some other types of functions. The concept of a pseudomonotone map was introduced by Karamardian [4]. He also showed that a differentiable function is pseudoconvex if and only if its gradient map is pseudomonotone. Later on Karamardian and Schaible [5] introduced some new types of monotone and generalized monotone maps. For all types but one, they established that a differentiabe function possesses some convexity type property if and only if its gradient map possesses the corresponding monotonicity type property. A question remained open in the case when the strong pseudoconvexity of the underlying function is equivalent to the strong pseudomonotonicity of its gradient map. After that Hadjisavvas and Schaible [2] introduced another notion of a strongly pseudomonotone map and they derived its correspondence with the concept of strong pseudoconvexity from the paper of Diewert, Avriel, Zang [1] in the twice differentiable case. They also obtained that this is not true for the notion of [5] by presenting a counterexample.

Now we introduce the concepts of very strong pseudoconvexity of a differentiable function and very strong pseudomonotonicity of a map. We show that the underlying function is very strongly pseudoconvex if and only if its gradient map is very strongly pseudomonotone. The notion of very strong pseudoconvexity is stronger than the concept of strong pseudoconvexity in [1] in the following sense. Each very strongly pseudoconvex function is strongly pseudoconvex in the sense of Diewert, Avriel and Zang [1]. In the same sense the notion of very strong pseudomonotonicity is stronger than the strong pseudomonotonicity in Hadjisavvas and Schaible [2].

In comparison with [2], to prove the correspondence between the very strong pseudoconvexity of f and the very strong pseudomonotonicity of ∇f, we don't need of twice differentiable function. It is sufficient the function to be only differentiable.

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Another notions of strong pseudoconvexity are considered in Mishra, Nanda, Acharya [6] and Weir [7].

2. Main result. In the sequel X is an open convex set in the finite-dimensional Euclidean space $\mathbb{R}^n$, and the function f is defined on X.

Definition 1. *A differentiable function f is said to be* very strongly pseudoconvex (*shortly, v.s.p.c.*) *on X if there exists a constant $\alpha > 0$ such that the following implication holds*

$$\text{(VSPC)} \qquad x \in X,\ y \in X,\ f(y) \le f(x) \implies (y-x)^T \nabla f(x) + \alpha \|y - x\| \le 0.$$

Recall the following well-known notions. The differentiable function f is said to be (strictly) pseudoconvex on X if for all (distinct) $x \in X$, $y \in X$

$$f(y) < f(x) \text{ (or } f(y) \le f(x)) \quad \text{implies} \quad (y-x)^T \nabla f(x) < 0.$$

It is obvious that each v.s.p.c. function is strictly pseudoconvex.

Example 1. Consider the function of one variable

$$f(x) = \frac{1}{x+1}, \quad x \in (0,1).$$

It is v.s.p.c. function. Indeed, $f(y) \le f(x)$ if and only if $y \ge x$. Therefore, if $\alpha \le \frac{1}{4}$, then the implication (VSPC) holds.

Example 2. The following example shows that the classes of v.s.p.c. and strictly pseudoconvex functions don't coincide. Consider the function of one variable

$$f(x) = \frac{1}{x+1}, \quad x \in (0,\infty).$$

It is strictly pseudoconvex, but this function is not v.s.p.c. Indeed, if

$$f(y) \le f(x),$$

then the inequality

$$(y-x)^T \nabla f(x) + \alpha \|y-x\| \le 0$$

holds if and only if $\alpha - \frac{1}{(x+1)^2} \le 0$. For all $\alpha > 0$ there exists $x > 0$ such that $\alpha > \frac{1}{(x+1)^2}$, and hence this function is not v.s.p.c.

Definition 2. *We call the map $F : X \to \mathbb{R}^n$* very strongly pseudomonotone (*shortly, v.s.p.m.*) *on X if there exists a constant $\beta > 0$ satisfying the following implication*

$$\text{(VSPM)} \qquad \begin{aligned} x \in X,\ y \in X,\ (y-x)^T F(x) + \beta \|y-x\| > 0 \\ \implies (x-y)^T F(y) < 0. \end{aligned}$$

Recall that the map $F : X \to \mathbb{R}^n$ is said to be strictly pseudomonotone on X [5] if for every pair of distinct points $x \in X$, $y \in X$, we have

$$(y-x)^T F(x) \ge 0 \quad \text{implies} \quad (x-y)^T F(y) < 0.$$

It is obvious that each v.s.p.m. map is strictly pseudomonotone.

Recall the following concept due to Hadjisavvas and Schaible [2].

Definiton 3. *A map $F : X \to \mathbb{R}^n$ is said to be* strongly pseudomonotone *on X if,* *for every $x \in X$ and $v \in \mathbb{R}^n$ such that $\|v\| = 1$, $v^T F(x) = 0$, there exist positive ε, $\bar{\beta}$ with $x \pm \varepsilon v \in X$ and*

$$v^T F(x + tv) \geq \bar{\beta} t, \quad 0 \leq t < \varepsilon \tag{1}$$

Proposition 1. *Let the map $F : X \to \mathbb{R}^n$ be v.s.p.m. in the sense of Definion 2. Then it is strongly pseudomonotone in the sense of Definition 3.*

Proof. Let there exist $\beta > 0$ such that the map F fulfils the implication (VSPM). Suppose that $v \in \mathbb{R}^n$ and $x \in X$ satisfy the equalities

$$\|v\| = 1 \text{ and } v^T F(x) = 0.$$

We choose $\varepsilon > 0$ and $\bar{\beta} > 0$ such that $\varepsilon \bar{\beta} \leq \beta$. It follows from the implication (VSPM) that

$$(tv)^T F(x + tv) \geq \beta \|tv\| \text{ for all } t \in [0, \varepsilon).$$

Therefore, $v^T F(x + tv) > \bar{\beta} t$ for all $t \in [0, \varepsilon)$. $\square$

The following two theorems say that the very strong pseudoconvexity of a function is equivalent to the very strong pseudomonotonicity of its gradient map.

Theorem 1. *If a differentiable function f is v.s.p.c. on X with a constant $\alpha > 0$, then its gradient map ∇f is v.s.p.m. on X with the same constant.*

Proof. Assume that f is v.s.p.c. with a constant α, and

$$x,\ y \in X,\ (y - x)^T \nabla f(x) + \alpha \|y - x\| > 0.$$

By implication (VSPC) $f(y) > f(x)$. It is obvious that the function f is pseudoconvex. It follows from pseudoconvexity that

$$(x - y)^T \nabla f(y) < 0,$$

which implies that ∇f is v.s.p.m. with a constant α. $\square$

Remark 1. In the corresponding statement of Hadjisavvas and Schaible [2, Proposition 3.5] the underlying function must be twice differentiable, and their assertion, in general, does not hold without this assumption.

Theorem 2. *If the function f is differentiable on X, and the gradient map ∇f is v.s.p.m. on X with a constant β, then f is v.s.p.c. on X with the same constant.*

Proof. Let ∇f be v.s.p.m. with a constant β. Assume that implication (VSPC) fails, that is for all $\alpha > 0$ there exist $x,\ y \in X$ such that $f(y) \leq f(x)$ and

$$(y - x)^T \nabla f(x) + \alpha \|y - x\| > 0.$$

In particular, these inequalities hold when $\alpha = \beta$. According to the Mean-Value Theorem there exist a $t \in (0, 1)$ such that

$$f(y) - f(x) = (y - x)^T \nabla f(z), \quad \text{where} \quad z = x + t(y - x).$$

Hence, $(y - x)^T \nabla f(z) \leq 0$. From the inequality

$$(y - x)^T \nabla f(x) + \beta \|y - x\| > 0$$

it follows that

$$(z - x)^T \nabla f(x) + \beta \|z - x\| > 0.$$

By implication (VSPM),

$$(x - z)^T \nabla f(z) < 0.$$

Therefore, $(y-x)^T\nabla f(z) > 0$, which is a contradiction. Consequently, f is v.s.p.c. $\square$

Remark 2. It follows from Proposition 1, Theorem 1, and Proposition 3.4 in [2] that our notion of strong pseudoconvexity is stronger than the corresponding one of [1].

The implication (VSPC) is equivalent to the following implication. There exist a constant $\gamma > 0$ such that

$$\begin{gathered} x \in X,\ y \in X,\ x \neq y,\ f(y) \le f(x) \\ \Longrightarrow\ (y-x)^T\nabla f(x) + \gamma\|y-x\| < 0. \end{gathered} \tag{2}$$

Indeed, we may take $\gamma = \dfrac{\alpha}{2}$.

The following theorems give complete characterizations of the very strong pseudoconvexity and very strong pseudomonotonicity.

Theorem 3. *A differentiable function f is v.s.p.c. on X if and only if there exists a constant γ such that for all distinct $x \in X$, $y \in X$ there exists a positive function $p(x,y)$ satisfying the inequality*

$$f(y) - f(x) > p(x,y)((y-x)^T\nabla f(x) + \gamma\|y-x\|) \tag{3}$$

Proof. *Sufficiency.* It follows directly from (3), since $p(x,y) > 0$.

Necessity. Let the implication (3) hold. Consider the function

$$p(x,y) = \begin{cases} 2\dfrac{f(y)-f(x)}{(y-x)^T\nabla f(x) + \gamma\|y-x\|}, & \text{if} \quad f(y) < f(x), \\ \dfrac{1}{2}\dfrac{f(y)-f(x)}{(y-x)^T\nabla f(x) + \gamma\|y-x\|}, & \text{if} \quad (y-x)^T\nabla f(x) + \gamma\|y-x\| > 0, \\ 1, & \text{otherwise}. \end{cases}$$

According to the very strong pseudoconvexity, the sets

$$\{(x,y) \in X \times X \mid f(y) < f(x)\}$$

and

$$\{(x,y) \in X \times X \mid (y-x)^T\nabla f(x) + \gamma\|y-x\| > 0\}$$

have an empty intersection. Consequently, p is well defined. If

$$f(y) < f(x) \text{ or } (y-x)^T\nabla f(x) + \gamma\|y-x\| > 0,$$

then, by (2), $p(x,y) > 0$, and the inequality (3) holds. It is seen that when

$$f(y) \ge f(x) \text{ and } (y-x)^T\nabla f(x) + \gamma\|y-x\| \le 0$$

the inequality (3) is satisfied again. $\square$

Theorem 4. *The map F is v.s.p.m. on X if and only if there exists a constant γ such that for all distinct $x \in X$, $y \in X$ there exists a positive function $p(x,y)$ satisfying the inequality*

$$(y-x)^T F(y) - \gamma\|y-x\| > p(x,y)(y-x)^T F(x).$$

Proof. It is similar to the proof of Theorem 3. We may consider the function

$$p(x,y)=\begin{cases}\dfrac{1}{2}\dfrac{(y-x)^TF(y)-\gamma\|y-x\|}{(y-x)^TF(x)}, & \text{if} \quad (y-x)^TF(x)>0,\\ 2\dfrac{(y-x)^TF(y)-\gamma\|y-x\|}{(y-x)^TF(x)}, & \text{if} \quad (y-x)^TF(y)-\gamma\|y-x\|<0,\\ 1, & \text{otherwise,}\end{cases}$$

where $\gamma>0$ is a constant which satisfies the implication

$$x,\, y\in X,\ x\neq y,\ (y-x)^TF(x)\geq 0\Rightarrow (x-y)^TF(y)+\gamma\|x-y\|<0. \qquad \square$$

The above characterizations of pseudoconvex functions, strictly pseudoconvex ones, pseudomonotone maps, and strictly pseudomonotone ones are similar to that of Theorems 3 and 4 obtained in [3].

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МНОГО СИЛНА ПСЕВДОИЗПЪКНАЛОСТ И МНОГО СИЛНА ПСЕВДОМОНОТОННОСТ

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В тази статия се въвеждат понятията много силно псевдоизпъкнала функция и много силно псевдомонотонно изображение. Доказано е, че една диференцируема функция е много силно псевдоизпъкнала тогава и само тогава, когато нейният градиент е много силно псевдоизпъкнало изображение. Получени са характеризации на много силно псевдоизпъкналите функции и много силно псевдомонотонните изображения.