МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2005
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2005
Proceedings of the Thirty Fourth Spring Conference of the Union of Bulgarian Mathematicians
Borovets, April 6–9, 2005

MEASURABILITY OF SETS OF PAIRS OF SPHERES AND LINES IN THE SIMPLY ISOTROPIC SPACE*

Margarita G. Spirova, Adrijan V. Borisov

The measurability under the group of the general simply isotropic similitudes of sets of pairs of spheres and straight lines is studied and some Crofton type formulas are given.

1. Introduction. The simply isotropic space ${I_3}^{(1)}$ is a projective space $\mathbb{P}_3(\mathbb{R})$ with an absolute plane ω and two complex conjugate straight lines f_1, f_2 in ω with a (real) intersection point F. All regular projectivities transforming the absolute figure into itself form the 8-parametric group G_8 of the general simply isotropic similitudes. In the affine coordinates (x, y, z) any similitude of G_8 can be written in the form

$$\begin{aligned} \overline{x} &= c_1 + c_7(x\cos\varphi - y\sin\varphi), \\ \overline{y} &= c_2 + c_7(x\sin\varphi + y\cos\varphi), \\ \overline{z} &= c_3 + c_4x + c_5y + c_6z, \end{aligned} \tag{1}$$

where $c_1, c_2, c_3, c_4, c_5, c_6 \neq 0, c_7 > 0$ and φ are real parameters. The more of the geometry of $I_3^{(1)}$ has been treated in detail in [4].

Using some basic concepts of the integral geometry in the sense of M. I. Stoka [5], G. I. Drinfel'd [3] we study the measurability of sets of pairs of spheres and straight lines in ${I_3}^{(1)}$ with respect to G_8.

2. Measurability of a set of pairs consisting of parabolic sphere and nonisotropic straight lines. Let (Σ, G) be a pair of the parabolic sphere [4;p.66]

$$\Sigma : x^2 + y^2 + 2\alpha x + 2\beta y + 2\gamma z + \delta = 0, \quad \gamma \neq 0 \tag{2}$$

and the nonisotropic straight line [4;p.5]

$$G : \; x = az + p, y = bz + q, \quad (a, b) \neq (0, 0), \tag{3}$$

where

$$[a(p+\alpha) + b(q+\beta) + \gamma]^2 - (a^2 + b^2)[(p+\alpha)^2 + (q+\beta)^2 + \delta - \alpha^2 - \beta^2] \neq 0, \tag{4}$$

i. e. G is not a tangent of Σ.

---

*2000 Mathematics Subject Classification:** 5365

Under the action of (1) the pair (Σ, G) $(\alpha, \beta, \gamma, \delta, a, b, p, q)$ is transformed into the pair $(\overline{\Sigma}, \overline{G})$ $(\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}, \overline{a}, \overline{b}, \overline{p}, \overline{q})$ according to

$$
\begin{aligned}
&\overline{\alpha} = -c_1 + c_6^{-1} c_7[(c_6\alpha - c_4\gamma)\cos\varphi - (c_6\beta - c_5\gamma)\sin\varphi],\\
&\overline{\beta} = -c_2 + c_6^{-1} c_7[(c_6\alpha - c_4\gamma)\sin\varphi + (c_6\beta - c_5\gamma)\cos\varphi],\\
&\overline{\gamma} = c_6^{-1} c_7^2 \gamma,\\
&\overline{\delta} = (c_1^2 + c_2^2)c_7^2\delta - 2c_6^{-1}c_7\{c_3c_7\gamma + [(c_6\alpha - c_4\gamma)c_1 + (c_6\beta - c_5\gamma)c_2]\cos\varphi -\\
&\qquad\qquad - [(c_6\beta - c_5\gamma)c_1 - (c_6\alpha - c_4\gamma)c_2]\sin\varphi\},\\
(5)\qquad &\overline{a} = Kc_7(a\cos\varphi - b\sin\varphi),\\
&\overline{b} = Kc_7(a\sin\varphi + bcos\varphi),\\
&\overline{p} = c_1 + Kc_7\{[-c_3a + c_5(bp - aq) + c_6p]\cos\varphi +\\
&\qquad + [c_3b + c_4(bp - aq) - c_6q]\sin\varphi\},\\
&\overline{q} = c_2 + Kc_7\{[-c_3a + c_5(bp - aq) + c_6p]\sin\varphi -\\
&\qquad - [c_3b + c_4(bp - aq) - c_6q]\cos\varphi\},
\end{aligned}
$$

where $K = (c_4a + c_5b + c_6)^{-1}$. The transformations (5) form so-called associated group $\overline{G_8}$ of G_8 [5;p.34]. $\overline{G_8}$ is isomorphic to G_8 and the invariant density under G_8 of the pairs (Σ, G), if it exists, coincides with the invariant density under $\overline{G_8}$ of the points $(\alpha, \beta, \gamma, \delta, a, b, p, q)$ in the set of parameters [5;p.33]. The associated group $\overline{G_8}$ has the infinitesimal operators

$$
\begin{aligned}
&Y_1 = \frac{\partial}{\partial\alpha} + 2\alpha\frac{\partial}{\partial\delta} - \frac{\partial}{\partial p},\ Y_2 = \frac{\partial}{\partial\beta} + 2\beta\frac{\partial}{\partial\delta} - \frac{\partial}{\partial q},\ Y_3 = 2\gamma\frac{\partial}{\partial\delta} + a\frac{\partial}{\partial p} + b\frac{\partial}{\partial q},\\
&Y_4 = \gamma\frac{\delta}{\delta\alpha} + a(a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b}) + p(a\frac{\partial}{\partial p} + b\frac{\partial}{\partial q}),\\
&Y_5 = \gamma\frac{\delta}{\delta\beta} + b(a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b}) + q(a\frac{\partial}{\partial p} + b\frac{\partial}{\partial q}),\ Y_6 = \gamma\frac{\partial}{\partial\gamma} + a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b},\\
&Y_7 = \alpha\frac{\partial}{\partial\alpha} + \beta\frac{\partial}{\partial\beta} + 2\gamma\frac{\partial}{\partial\gamma} + 2\delta\frac{\partial}{\partial\delta} + a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b} + p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q},\\
&Y_8 = -\beta\frac{\partial}{\partial\alpha} + \alpha\frac{\partial}{\partial\beta} - b\frac{\partial}{\partial a} + a\frac{\partial}{\partial b} - q\frac{\partial}{\partial p} + p\frac{\partial}{\partial q}.
\end{aligned}
$$

The integral invariant function $f = f(\alpha, \beta, \gamma, \delta, a, b, p, q)$ satisfies the system of R. Deltheil [2;p.28]

$$Y_1(f) = 0,\ Y_2(f) = 0,\ Y_3(f) = 0,\ Y_4(f) + 4af = 0,\ Y_5(f) + 4bf = 0,$$

$$Y_6(f) + 3f = 0,\ Y_7 + 10f = 0,\ Y_8(f) = 0$$

and has the form

$$f = \frac{c(a^2 + b^2)\gamma}{\{[a(p+\alpha) + b(q+\beta) + \gamma]^2 - (a^2 + b^2)[(p+\alpha)^2 + (q+\beta)^2 + \delta - \alpha^2 - \beta^2]\}^4}$$

where $c = const \neq 0$.

Choosing $c = 1$, we have

Theorem 1. *The invariant density under G_8 for the pairs (Σ, G) satisfying* (2), (3) *and* (4) *is*

$$d(\Sigma, G) =$$

(6)
$$= \frac{(a^2+b^2)|\gamma|\, d\alpha\wedge d\beta\wedge d\gamma\wedge d\delta\wedge da\wedge db\wedge dp\wedge dq}{\{[a(p+\alpha)+b(q+\beta)+\gamma]^2-(a^2+b^2)[(p+\alpha)^2+(q+\beta)^2+\delta-\alpha^2-\beta^2]\}^4}.$$

3. Some Crofton type formulas. Now we are going to express the 4-form $d\alpha\wedge d\beta\wedge d\gamma\wedge d\delta$ in the terms of the radius R and the vertex $Q(x_0, y_0, z_0)$ of the parabolic sphere Σ. From

(7)
$$R = -\frac{1}{2\gamma},\quad x_0 = -\alpha,\quad y_0 = -\beta,\quad z_0 = -\frac{\delta-\alpha^2-\beta^2}{2\gamma}$$

we have

$$dR = \frac{1}{2\gamma^2}\, d\gamma,\quad dx_0 = -d\alpha,\quad dy_0 = -d\beta,$$
$$dz_0 = \frac{\alpha}{\gamma}\, d\alpha + \frac{\beta}{\gamma}\, d\beta + \frac{\delta-\alpha^2-\beta^2}{2\gamma^2}\, d\gamma - \frac{1}{2\gamma}\, d\delta$$

and by exterior multiplication we get

(8)
$$d\alpha\wedge d\beta\wedge d\gamma\wedge d\delta = -\frac{1}{2R^3}\, dx_0\wedge dy_0\wedge dz_0\wedge dR.$$

From (6), (7) and (8) it follows that

$$d(\Sigma, G) =$$

(9)
$$= \frac{64R^4(a^2+b^2)^2\, dx_0\wedge dy_0\wedge dz_0\wedge dR\wedge da\wedge db\wedge dp\wedge dq}{\{[2aR(p-x_0)+2bR(q-y_0)-1]^2-4(a^2+b^2)R[R(p-x_0)^2+R(q-y_0)^2+z_0]\}^4}.$$

On the other hand, the densities for the points $Q(x_0, y_0, z_0)$ and the straight lines $G(a, b, p, q)$ with respect the group $B_6^{(1)}$ of the simply isotropic motions in $I_3^{(1)}$ are [1]

$$dQ = dx_0\wedge dy_0\wedge dz_0,\quad dG = \frac{da\wedge db\wedge dp\wedge dq}{(a^2+b^2)^2},$$

respectively. Then the density (9) becomes

$$d(\Sigma, G) =$$

(10)
$$= \frac{64R^4(a^2+b^2)^2\, dQ\wedge dR\wedge dG}{\{[2aR(p-x_0)+2bR(q-y_0)-1]^2-4(a^2+b^2)R[R(p-x_0)^2+R(q-y_0)^2+z_0]\}^4}$$

Suppose that

(11)
$$[a(p+\alpha)+b(q+\beta)+\gamma]^2-(a^2+b^2)[(p+\alpha)^2+(q+\beta)^2+\delta-\alpha^2-\beta^2] > 0$$

and denote by P_1 and P_2 the intersection points of G and Σ. Then the distance $|P_1P_2|$ between nonparallel points P_1 and P_2 is

$$|P_1P_2| =$$
$$= 2\sqrt{\frac{[a(p+\alpha)+b(q+\beta)+\gamma]^2-(a^2+b^2)[(p+\alpha)^2+(q+\beta)^2+\delta-\alpha^2-\beta^2]}{a^2+b^2}}$$

and therefore

$$|P_1P_2| =$$

(12)
$$=\sqrt{\frac{[2aR(p-x_0)+2bR(q-y_0)-1]^2-4(a^2+b^2)R[R(p-x_0)^2+R(q-y_0)^2+z_0]}{R^2(a^2+b^2)}}.$$

From(10), applying (12), we obtain

(13)
$$d(\Sigma, G) = \frac{64}{R^4|P_1P_2|^8}\, dQ \wedge dR \wedge dG.$$

We summarize the foregoing results in the following

Theorem 2. *The density under G_8 for the pairs (Σ, G) determined by* (2), (3) *and* (4) *satisfies the relations* (9) *and* (10). *If* (11) *holds, then the density for the pairs (Σ, G) can be written in the form* (13).

4. Measurability of a set of pairs consisting of parabolic sphere and isotropic straight line. Let (Σ, G) be a pair of the parabolic sphere Σ determined by (2) and the isotropic straight line [4;p.5]

(14)
$$G:\ x = p, y = q.$$

Now the associated group $\overline{G_8}$ of the G_8 has the infinitesimal operators

$$Y_1 = \frac{\partial}{\partial \alpha} + 2\alpha\frac{\partial}{\partial \delta} - \frac{\partial}{\partial p},\ Y_2 = \frac{\partial}{\partial \beta} + 2\beta\frac{\partial}{\partial \delta} - \frac{\partial}{\partial q},$$
$$Y_3 = 2\gamma\frac{\partial}{\partial \delta},\ Y_4 = \gamma\frac{\partial}{\partial \alpha},\ Y_5 = \gamma\frac{\partial}{\partial \beta},\ Y_6 = \gamma\frac{\partial}{\partial \gamma},$$
$$Y_7 = \alpha\frac{\partial}{\partial \alpha} + \beta\frac{\partial}{\partial \beta} + 2\gamma\frac{\partial}{\partial \gamma} + 2\delta\frac{\partial}{\partial \delta} + p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q},$$
$$Y_8 = -\beta\frac{\partial}{\partial \alpha} + \alpha\frac{\partial}{\partial \beta} - q\frac{\partial}{\partial p} + p\frac{\partial}{\partial q}$$

and Y_1, Y_2, Y_3, Y_4, Y_5 and Y_6 are unconnected but

$$Y_7 = -pY_1 - qY_2 + \frac{\alpha p + \beta q + \delta}{\gamma}\, Y_3 + \frac{p+\alpha}{\gamma}\, Y_4 + \frac{q+\beta}{\gamma}\, Y_5 + 2Y_8.$$

Since

$$Y_1(-p) + Y_2(-q) + Y_3(\frac{\alpha p + \beta q + \delta}{\gamma}) + Y_4(\frac{p+\alpha}{\gamma}) + Y_5(\frac{q+\beta}{\gamma}) + Y_8(2) \neq 0$$

it follows [3] that we can state:

Theorem 3. *A set of pairs (Σ, G) of a parabolic sphere Σ and an isotropic line G is not measurable and it has not measurable subsets with respect to the group G_8.*

5. Measurability of a set of pairs consisting of cylindrical sphere and nonisotropic straight line. Let (Σ, G) be a pair of the cylindrical sphere [4; p.66]

(15)
$$\Sigma : x^2 + y^2 + 2\alpha x + 2\beta y + \delta = 0, \quad \alpha^2 + \beta^2 - \delta > 0$$

and the nonisotropic straight line G determined by (3). By arguments similar to the ones used in the section 4, we find:

Theorem 4. *A set of pairs (Σ, G) of a cylindrical sphere Σ and a nonisotropic straight line G is not measurable and it has not measurable subsets with respect to the group G_8.*

6. Measurability of a set of pairs consisting of cylindrical sphere and isotropic line. Let (Σ, G) be a pair of the cylindrical sphere Σ and the isotropic straight line G determined by (14) and (15), respectively. If

$$p^2 + q^2 + 2\alpha p + 2\beta q + \delta \neq 0, \tag{16}$$

i.e. G is not an element of Σ, then we have the following

Theorem 5. *With respect to the group G_8 a set of pairs (Σ, G) of the cylindrical sphere Σ and the isotropic straight line G satisfying* (14), (15) *and* (16) *is not measurable but has the measurable subset*

$$(p+\alpha)^2 + (q+\beta)^2 = h(\alpha^2 + \beta^2 - \delta), \quad h > 0, h \neq 0, 1$$

with the density

$$d(\Sigma, G) = \frac{d\alpha \wedge d\beta \wedge dp \wedge dq}{[(p+\alpha)^2 + (q+\beta)^2]^2}.$$

REFERENCES

[1] A. Borisov, M. Spirova. Some integral geometric results in the simply isotropic space. *J. geometry*, **77** (2003), 8–21.

[2] R. Deltheil. Sur la théorie des probabilité géométriques. *Thése Ann. Fac. Sc. Univ. Toulouse*, (3), **11** (1919), 1–65.

[3] G. I. Drinfel'd. On the measure of the Lie groups. *Zap. Mat. Otdel. Fiz. Mat. Fak. Kharkov. Mat. Obsc.*, **21** (1949), 47–57 (Russian).

[4] H. Sachs. Isotrope Geometrie des Raumes. Friedr. Vieweg and Sohn, Braunschweig/Wiesbaden, 1990.

[5] M. I. Stoka. Geometrie Integrală. Ed. Acad. RPR, Bucuresti, 1967.

Margarita Georgieva Spirova
Dept. of Descriptive Geometry
Univ. of Architecture, Civil Eng. and Geodesy
1, Christo Smirnenski Blvd.
1046 Sofia, Bulgaria
e-mail: spirov_fgs@uacg.bg

Adrijan Varbanov Borisov
Dept. of Mathematics
Neofit Rilski South-West University
66, Ivan Mihailov Str.
2700 Blagoevgrad, Bulgaria
e-mail: adribor@aix.swu.bg

ИЗМЕРИМОСТ НА МНОЖЕСТВА ОТ ДВОЙКИ СФЕРИ И ПРАВИ В ПРОСТО ИЗОТРОПНО ПРОСТРАНСТВО

Маргарита Г. Спирова, Адриян В. Борисов

Изследвана е измеримостта на множества от двойки сфери и прави относно групата на общите просто-изотропни подобности и са получени някои формули от крофтонов тип.