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A LOWER BOUND FOR THE DIMENSION DIAMETERS OF CERTAIN SETS WITH RESPECT TO ESSENTIAL SYSTEMS

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Let $\mathcal{E} = \{(A_i, B_i)\}$, $i = 1, \dots, n$ be an essential system in the normal space X. We prove in this paper, that if $\mathcal{U}$ is a finite open covering of X and $\operatorname{ord}\mathcal{U} \geq n$, then some element of $\mathcal{U}$ intersects two opposite faces of $\mathcal{U}$. Various consequences of this result are discussed.

1. Basic concepts and definitions. Let $\mathcal{U}$ be an open covering of the normal space X and $Y \subset X$. We say that the n-dimensional diameter $d_n(Y)$ of Y is greater than $\mathcal{U}$, (the notation is $d_n(Y) \geq \mathcal{U}$), if the order of every refinement $\mathcal{V}$ of the restriction $\mathcal{U}_Y$ is greater than $n+2$: $\operatorname{ord}\mathcal{V} \geq n+2$.

If X is a metric space, then the n-dimensional diameter $d_n(Y)$ of the subset Y is the number $\inf\{\operatorname{mesh}(\mathcal{U}_Y)\}$, where $\mathcal{U}$ runs the set of all open coverings of X with $\operatorname{ord}(\mathcal{U}_Y) \leq n+1$.

Note that some authors refer to d_n as d_{n+1} ([2], [1]). In this papers d_n is called an n-dimensional degree. Here we follow the terminology which is adopted in [3].

For a metric space (X, ϱ), the inequality $d_n(X) > 0$ means that the metric dimension μ-$\dim X$ of X is not less than $n+1$ [5]. Clearly, for compact metric spaces, $d_n(X) = 0$ if and only if $\dim X \leq n$.

Definition 1.1. [3] *Let n be an integer and $\mathcal{U}$ be an open covering of X. The normal space X is referred as $(n,\mathcal{U})$-connected between the closed sets P and Q if $d_{n-2}(C) > \mathcal{U}$ for an arbitrary partition C between P and Q in X.*

In the case of metric space (X, ρ) we shall say that X is (n, ε)-connected between P and Q if $d_{n-2}(C) \geq \varepsilon$ (clearly, here $\varepsilon > 0$).

Furthermore, suppose that the system $\mathcal{E} = \{(A_1, B_1); (A_2, B_2); \dots; (A_n, B_n)\}$ consists of n disjoint pairs of closed subsets of X.

Definition 1.2. *The system $\mathcal{E}$ is essential (or n-defining [5]) if for any closed sets P_i, $i = 1, \dots, n$, separating A_i and B_i (partitions), the intersection $\bigcap_{i=1}^{n} P_i$ is nonempty.*

Obviously, the essential system is an analogue of the system of opposite faces of the $n-$dimensional cube I^n; $I = [0, 1]$. Because of that, we call A_i and B_i faces of $\mathcal{E}$. An important tool in this note is the following Lemma, which seems to have an independent meaning.

Lemma 1.3. *Let $\mathcal{E}$ be an essential system in the normal space X and $\mathcal{U}$ be a locally finite open covering of X with* $\operatorname{ord} \mathcal{U} \leq n$. *Then, some element of $\mathcal{U}$ intersects two opposite faces of $\mathcal{E}$.*

Proof. Suppose that no element of $\mathcal{U}$ intersects opposite faces of $\mathcal{E}$. According to [4] (p. 119, Assertion 8), $\mathcal{U}$ can be refined by a closed n-discrete locally finite covering $\mathcal{C}$ of X. In other words there is an n discrete locally finite refinement $\mathcal{C}$ of $\mathcal{U}$

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_n,$$

where $\mathcal{C}_i$ is a discrete family of closed subsets of X for every i. Let us put for every i $(1 \leq i \leq n)$:

$$A_i' = A_i \cup \bigcup \{C \in \mathcal{C}_i | C \cap A_i \neq \emptyset\} \text{ and } B_i' = A_i \cup \bigcup \{C \in \mathcal{C}_i | C \cap A_i = \emptyset\}.$$

According to our hypothesis, the sets A_i' and B_i' are disjoint (and obviously, closed). The space X is normal, hence for every i one can find a closed partition P_i between A_i' and B_i' in X. Clearly, P_i is at the same time a partition between A_i and B_i because $A_i \subset A_i'$ and $B_i \subset B_i'$. Since the system $\mathcal{E}$ is essential, one has $P_0 = \bigcap\limits_{i=1}^{n} P_i \neq \emptyset$. On the other hand, this is impossible, since $P_0 \cap \bigcup \mathcal{C}_i = \emptyset$ for every i. Thus we have obtained the desired contradiction. The above lemma proves also the proposition, stated in the abstract. In what follows some definitions and applications of the preceding lemma are present.

Let X be a normal space with $\dim X \geq n$. Then X contains an essential n-system $\mathcal{E}$ (the Eilenberg - Otto characterization of dimension). Next, to every index i we associate an open covering $\mathcal{U}_i = \{X \setminus A_i, X \setminus B_i\}$.

Corollary 1.4. *Let $\mathcal{U}_i = \{X \setminus A_i, X \setminus B_i\}$ and $\mathcal{V} = \bigwedge\limits_{i=1}^{n} \mathcal{U}_i$ be the intersection of the coverings $\{\mathcal{U}_i\}$. Then the space X is $(n, \mathcal{V})$-connected between every pair (A_i, B_i) of opposite faces of $\mathcal{E}$.*

Proof. We confine ourselves to the case $i = 1$. Note that if P is a partition between A_1 and B_1, then the restriction $\mathcal{E}_P = \{(A_2 \cap P, B_2 \cap P); \ldots; (A_n \cap P, B_n \cap P)$ is an essential $(n-1)-$system in the normal space $P \cup \bigcup\limits_{i=2}^{n} (A_i \cup B_i)$.

Now, suppose that (X, ρ) is a metric space, and let

$$\mathcal{E} = \{(A_1, B_1); (A_2, B_2); \ldots; (A_n, B_n)\}$$

be an essential system in X for which $\delta_i = \rho(A_i, B_i) > 0$, $i = 1, \ldots, n$. If P is a partition between some pair (A_i, B_i), then $d_{n-2}(P) \geq \min\limits_i \{\delta_i\}$.

Definition 1.5. *The subset L of X is cutting X between the closed subsets P and Q, if for every closed subset Y of X, which is connected between $Y \cap P$ and $Y \cap Q$, we have $Y \cap L \neq \emptyset$.*

Next we use the following result from [7]:

Lemma 1.6. *Suppose that L cuts the normal space X between the disjoint closed nonempty sets P and Q. Then every open neighborhood O of L contains a (closed in X) partition C between P and Q.*

Using the preceding Lemma makes evident that the following assertion holds true:

Corollary 1.7. *If $d_{n-2}(L) < \mathcal{V}$, then L cuts X between A_i and B_i for no i.*

2. The n-dimensional cube. Next, suppose that (X, ρ) is a metric space. For the following theorem suppose also that $X = \bigcup_{i=1}^{\infty} F_i$, where F_i is a closed subset of X for every i.

Theorem 2.1. *If X is a compact $n-$dimensional space, then there are constants $\varepsilon_X > 0$ and $\delta_X > 0$ such that if $d_{n-1}(F_i) < \varepsilon_X$ for every i, then the inequality*

$$d_{n-2}\left(\bigcup_{i \neq j}(F_i \cap F_j)\right) \geq \delta_X$$

holds. Hence, $\mu - \dim\left(\bigcup_{i \neq j}(F_i \cap F_j)\right) \geq n-1$.

Proof. Let $\mathcal{E} = \{(A_1, B_1); (A_2, B_2); \ldots; (A_n, B_n)\}$ be an essential system in X and put $\varepsilon_X = \max_i \rho(A_i, B_i) = \rho(A_{i_0}, B_{i_0})$. Then no set F_i intersects both A_{i_0} and B_{i_0}. On the other hand, X is (n, δ_X)-connected between A_{i_0} and B_{i_0}. Now, suppose that for $M = \bigcup_{i \neq j}(F_i \cap F_j)$ the inequality $d_{n-2} < \delta_X$ holds. Then, $X \setminus M$ is connected between A_{i_0} and B_{i_0} and, hence, the complement of M contains a continuum K, which connects the faces in question. Thus $K = \bigcup_{i=1}^{\infty}(K \cap F_i)$ and the summands are proper closed subsets of K, which contradicts to the well known Sierpinski theorem [6].

There are various consequences of Lemma 1.3. We shall confine ourselves to the question of calculating the dimensional diameters of the $n-$ dimensional unit cube $I^n = [0,1]^n$ and to establish the degree of connectedness of I^n. We consider the $n-$dimensional cube I^n equipped with the "maximum" metric $r(x,y) = \max|y_i - x_i|$. The corresponding dimensional diameter will be denoted by d_k^r.

Theorem 2.2. $d_k^r(I^n) = 1$ *for every* $k = 1, \ldots, n-1$.

Proof. The diameter of I^n under the "maximum" metric is equal to 1. Hence, $d_0^r(I^n) = 1$. On the other hand, by Lemma 1.4 it follows directly that $d_{n-1}(I^n) = 1$. Now, it remains to note that for every k we have $d_0(X) \leq d_k(X) \leq d_{n-1}(X)$ (under arbitrary metric and for every space X).

Theorem 2.3. *Suppose that F_i for every i is a proper closed subset of I^n and that $I^n = \bigcup_{i=1}^{\infty} F_i$. Then $\mu - \dim\left(\bigcup_{i \neq j}(F_i \cap F_j)\right) \geq n-1$.*

Proof. The sets F_i are proper and, hence, at least two of them have non-void interiors. Clearly, one may suppose that these sets are F_1 and F_2. Let $Q_1 \subset F_1 \setminus F_2$ and $Q_2 \subset F_2 \setminus F_1$ be two subsets, which are homeomorphic copies of I^{n-1} and are parallel for example to A_1 (and B_1). It is easy to see that we may include Q_1 and Q_2 as opposite faces of some $n-$defining system. It follows from Corollary 1.7 that for some $\varepsilon > 0$, I^n is $(n, \varepsilon)-$connected between Q_1 and Q_2. Now, put $M = \bigcup_{i \neq j}(F_i \cap F_j)$ and then suppose that $d_{n-2}(M) < \varepsilon$. As in Theorem 2.1, one arrives to a contradiction to the Sierpinski theorem. Hence $d_{n-2}M \geq \varepsilon$, which implies $\mu - \dim M \geq n-1$.

Theorem 2.3 (under the hypothesis $\dim\left(\bigcup_{i\neq j}(F_i \cap F_j)\right) \geq n-1$) is contained in [8].

REFERENCES

[1] P. S. Urysohn. Memoire sur les multiplicites Cantoriennes (I), *Fundam. Math.*, 1925, **7**, 30–139.

[2] Y. Sternfeld. Extesion of mappings of Bing spaces into ANRs, *Topology and Appl.*, (1996), **80**, 189–194.

[3] P. S. Alexandroff. Die Kontinua (V^p) – eine Verschärfung der Cantorshen Mannigfaltigkeiten, *Monatsh. Math.*, Bd. **61**, H. 1, (1957), 67–76.

[4] P. Alexandroff, B. Pasinkov. An introduction to the Dimension Theory, (1973), Moscow, Nauka (in Russian).

[5] K. Nagami. Dimension Theory, (1970), Academic Press, New York and London.

[6] W. Sierpinski. Sur les ensenbles connexes et non connexes, *Fundam. Math.*, (1921), **2**, 81–95.

[7] V. T. Todorov. Decompositions of finite dimensional compacta, *Compt. rend. Acad. bulg. Sci.*, (to apper).

[8] N. G. Hadjiivanov. The n-dimensional cube can not be represented as a sum of countable many proper closed sets which pair-wise intersections are no more than $(n-2)$-dimensional, *Compt. Rend. Acad. Sci. of USSR*, **195**, No 1, (1970), 43–45 (in Russian).

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ДОЛНА ГРАНИЦА НА РАЗМЕРНОСТНИТЕ ДИАМЕТРИ НА НЯКОИ ПОДМНОЖЕСТВА НА СЪЩЕСТВЕНИ СИСТЕМИ

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Нека $\mathcal{E} = \{(A_i, B_i)\}$ е съществена система в нормалното пространство X. Всяка дизюнктна двойка (A_i, B_i) от срещуположните стени на $\mathcal{U}$ поражда отворено покритие $\mathcal{U} = \{X \setminus A_i, X \setminus B_i\}$. В тази бележка доказваме, че за някое i_0, $(n-1)$-мерният диаметър на $d_{n-1}(X)$ е по-голям от $\mathcal{U}_{i_0}$. Обсъждат се също така различни следствия от този резултат.