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THE DENSITY OF LINEAR TRANSFORMATIONS OF INDEPENDENT EXPONENTIAL RANDOM VARIABLES*

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The density function of linear transformations of independent exponential distributed random variables is derived.

Linear transformations of independent exponentially distributed random variables arise in many contexts in statistics and probability, for example as the distribution of quadratic forms $X'AX$ where X is an n-dimensional normal random vector and A is a suitable $n \times n$ positive semidefinite matrix. Such a quadratic form occurs, for example, as the limiting distribution of the chi-squared goodness-of-fit statistic when parameters are estimated on the basis of the ungrouped, rather than grouped, data – cf. Chernoff and Lehmann [1954]. Weighted sums of exponential random variables also occur in the form $-\log(\prod\limits_i P_i^{\theta_i})$, a weighted version of the Fisher statistic for combining independent p-values $P_1, \dots, P_n$, where each P_i is uniformly distributed on $(0,1)$ under the combined null hypothesis – cf. Good [1955] and Zelen and Joel [1959]. See Bock, Diaconis, Huffer and Perlman [1987] for additional examples.

Let $t^0, t^1, \dots, t^n$ be $n+1$ points in Euclidian space R_s, i.e. $t^i \equiv (t_1^i, \dots, t_s^i)$, $i = 0, \dots, n$. We suppose $n > s$ and that the points $t^0, t^1, \dots, t^n$ are "in general position". This means for $j = 1, \dots, s$ and all different indices $0 \le i_1, \dots, i_{j+1} \le n$ the determinants

(1) $$\begin{vmatrix} 1 & t_1^{i_1} & \dots & t_j^{i_1} \\ 1 & t_1^{i_2} & \dots & t_j^{i_2} \\ \dots & \dots & \dots & \dots \\ 1 & t_1^{i_{j+1}} & \dots & t_j^{i_{j+1}} \end{vmatrix} \neq 0.$$

The main goal of the authors is to derive the density function of the random vector

(2) $$\eta = \xi_1 t^1 + \dots + \xi_n t^n,$$

where ξ_i, $i = 0, 1, \dots, n$ are independent and identically distributed random variables with exponential density $f_{\xi_i}(x) = e^{-x}$, $x > 0$.

Further we will suppose $t^0 = 0^* \equiv (0, \dots, 0) \in R_s$ and we can represent η as

(3) $$\eta = \xi_0 t^0 + \xi_1 t^1 + \dots + \xi_n t^n.$$

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If we denote by

$$d(t^{i_1},\dots,t^{i_j},x;\gamma) := \begin{vmatrix} 1 & t_1^{i_1} & \dots & t_j^{i_1} \\ \dots & \dots & \dots & \dots \\ 1 & t_1^{i_j} & \dots & t_j^{i_j} \\ \gamma & x_1 & \dots & x_j \end{vmatrix}. \tag{4}$$

for $j = 1, 2, \dots, s$ and for $j = 0$ $d(x;\gamma) \equiv 1$ then conditions (1) can be expressed as $d(t^{i_1},\dots,t^{i_j},t^{i_{j+1}};1) \neq 0$.

Introduce for $j = 1,\dots,n$

$$a(t^{i_1},\dots,t^{i_j},x) = 0 \ \text{ if } \ d(t^{i_1},\dots,t^{i_j},0^*;1)/d(t^{i_1},\dots,t^{i_j};1) \leq 0$$

and

$$a(t^{i_1},\dots,t^{i_j},x) = -d(t^{i_1},\dots,t^{i_j},x;0)/d(t^{i_1},\dots,t^{i_j},0^*;1)$$

otherwise.

Let also introduce for $j = 1,\dots,s$:

$c(t^{i_1},\dots,t^{i_j},x) = -d(t^{i_1},\dots,t^{i_j},x;0)/d(t^{i_1},\dots,t^{i_j},0^*;1)$ if
$d(t^{i_1},\dots,t^{i_j},0^*;1)/d(t^{i_1},\dots,t^{i_j};1) < 0$; $c(t^{i_1},\dots,t^{i_j},x) = \infty$ if
$d(t^{i_1},\dots,t^{i_j},0^*;1)/d(t^{i_1},\dots,t^{i_j};1) > 0$; $c(t^{i_1},\dots,t^{i_j},x) = 0$ if
$d(t^{i_1},\dots,t^{i_j},0^*;1) = 0$
and $d(t^{i_1},\dots,t^{i_j},x;0)/d(t^{i_1},\dots,t^{i_j};1) < 0$; $c(t^{i_1},\dots,t^{i_j},x) = \infty$ if
$d(t^{i_1},\dots,t^{i_j},0^*;1) = 0$ and $d(t^{i_1},\dots,t^{i_j},x;0)/d(t^{i_1},\dots,t^{i_j};1) \geq 0$;
$A(t^{i_1},\dots,t^{i_s},x) = \max\big(0, a(t^{i_1},x), a(t^{i_1},t^{i_2},x),\dots,a(t^{i_1},\dots,t^{i_s},x)\big)$, $j = 1,2,\dots,s$;
$B(t^{i_1},\dots,t^{i_s},x) = \max\{A(t^{i_1},\dots,t^{i_s},x),\ C(t^{i_1},\dots,t^{i_s},x)\}$
$C(t^{i_1},\dots,t^{i_s},x) = \min\big(0, c(t^{i_1},x), c(t^{i_1},t^{i_2},,x),\dots,c(t^{i_1},\dots,t^{i_s},x)\big)$.

Theorem. *The density of the random vector* $\eta = \xi_1 t^1 + \dots + \xi_n t^n$ *at the point* $x \in R_s$, *is*

$$f_\eta(x) = \sum_{i_1=0}^{n} \sum_{\substack{i_2=0\\ i_2\neq i_1}}^{n} \cdots \sum_{\substack{i_s=0\\ i_s\neq i_j\\ j=1,\dots,s-1}} \frac{1}{\prod\limits_{\substack{i_{s+1}=0\\ i_{s+1}\neq ij\\ j=1,\dots,s}}^{n} d(t^{i_1},\dots,t^{i_{s+1}};1)} \times$$

$$\times \sum_{j=0}^{n-s} \frac{(d(t^{i_{1_1}},\dots,t^{i_s},0^*;1))^j (d(t^{i_1},\dots,t^{i_s},x;0))^{n-s-j}}{(n-s-j)!} \times$$

$$\times \sum_{l=0}^{j} \frac{(A(t^{i_1},\dots,t^{i_s},x))^l e^{-A(t^{i_1},\dots,t^{i_s},x)} - (B(t^{i_1},\dots,t^{i_s},x))^l e^{-B(t^{i_1},\dots,t^{i_s},x)}}{l!},$$

where $t^0 \equiv 0^*$, t^1, $\dots$, t^n *satisfy* (1)*;* $0^0 \equiv 1$ *and* $(+\infty)^m e^{-\infty} \equiv 0$ (*here* m *is non-negative integer*).

Proof. It is well known that the random vector η from (3) has the density function with respect to the s-dimensional Lesbegue measure if the rank of t^0, t^1, $\dots$, t^n is s. Under the conditions (1) the rank of t^0, t^1, $\dots$, t^n is s.

Representing

$$(5)\quad \eta=\left(\frac{\xi_0}{\xi_0+\cdots+\xi_n}.0^*+\frac{\xi_1}{\xi_0+\cdots+\xi_n}.t^1+\cdots+\frac{\xi_n}{\xi_0+\cdots+\xi_n}.t^n)(\xi_0+\cdots+\xi_n\right)$$

and using the result of Sukhatme [1937], we can assert that the random variable $\tau := \xi_0+\cdots+\xi_n$ and the random vector

$$\nu := \left(\frac{\xi_0}{\xi_0+\cdots\xi_n},\dots,\frac{\xi_n}{\xi_0+\cdots+\xi_n}\right)$$

are independent. Random variable τ is Gamma distributed with n-degree of freedom, i.e. with density

$$f_\tau(y)=\begin{cases}\dfrac{y^n e^{-y}}{n!}, & y>0\\ 0 & y\le 0\end{cases},$$

The random vector ν=$(\nu_0,\nu_1,\dots,\nu_n)$ has Dirichlet distribution with all parameters equal to 1 ($\nu\in D(1,1,\dots,1)$.

Using the formula of Ali and Mead [1968] for the density $f_\mu(x)$ of the random vector

$$\mu=(\frac{\xi_0}{\xi_0+\cdots+\xi_n}.0^*+\frac{\xi_1}{\xi_0+\cdots+\xi_n}.t^1+\cdots+\frac{\xi_n}{\xi_0+\cdots+\xi_n}.t^n)$$

we have under the conditions (1)

$$f_\mu(x)=\frac{n!}{(n-s)!}\sum_{i_1=0}^{n}\sum_{\substack{i_2=0\\ i_2\ne i_1}}^{n}\cdots\sum_{\substack{i_s=0\\ i_s\ne i_j\\ j=1,\dots,s-1}}^{n}\prod_{k=1}^{s}\nu(t^{i_1},\dots,t^{i_k},x)\times\frac{(d(t^{i_1},\dots,t^{i_s},x;1))^{n-s}}{\prod\limits_{\substack{i_{s+1}=0\\ i_{s+1}\ne i_j\\ j=1,\dots,s}}^{n} d(t^{i_1},\dots,t^{i_s},t^{i_{s+1}};1)},$$

where $\nu(t^{i_1},\dots,t^{i_k},x)=w(y(t^{i_1},\dots,t^{i_k},x))$, $k=1,2,\dots,s$;

$$w(n)=\begin{cases}1, & n\ge 0\\ 0, & n<0\end{cases}$$

and

$$y(t^{i_1},\dots,t^{i_k},x)=\frac{d(t^{i_1},\dots,t^{i_k},x;1)}{d(t^{i_1},\dots,t^{i_k};1)},\quad k=1,2,\dots,s.$$

From $\eta=\tau.\mu$ we get

$$f_\eta(x)=\int_0^\infty f_\tau(\theta).f_{\theta.\mu}(x)d\theta=\int_0^\infty\frac{\theta^n}{n!}e^{-\theta}f_{\theta.\mu}(x)d\theta$$

It is easy to find $f_{\theta.\mu}(x)=\dfrac{1}{\theta^s}f_\mu(\dfrac{1}{\theta}.x)$, $\theta\in(0,+\infty)$, x$\in R_s$.

Consequently

$$f_\eta(x)=\int_0^\infty\frac{\theta^n}{n!}e^{-\theta}.\frac{1}{\theta^s}f_\mu(\frac{1}{\theta}.x)d\theta=$$

$$=\int\limits_0^\infty \frac{\theta^{n-s}}{n!}e^{-\theta}\cdot\frac{n!}{(n-s)!}\sum_{i_1=0}^{n}\sum_{\substack{i_2=0\\ i_2\neq i_1}}^{n}\cdots\sum_{\substack{i_s=0\\ i_s\neq i_j\\ j=1,..,s-1}}^{n}\prod_{k=1}^{s}\nu(t^{i_1},\ldots,t^{i_k},\frac{1}{\theta}x))=\frac{(d(t^{i_1},..,t^{i_s},\frac{1}{\theta}x;1))^{n-s}}{\prod\limits_{\substack{i_{s+1}=0\\ i_{s+1}\neq i_j\\ j=1,..,s}}^{n} d(t^{i_1},..,t^{i_s},t^{i_{s+1}};1)}d\theta=$$

$$=\sum_{i_1=0}^{n}\sum_{\substack{i_2=0\\ i_2\neq i_1}}^{n}\cdots\sum_{\substack{i_s=0\\ i_s\neq i_j\\ j=1,\ldots,s-1}}^{n}\frac{1}{\prod\limits_{\substack{i_{s+1}=0\\ i_{s+1}\neq i_j\\ j=1,\ldots,s}}^{n} d(t^{i_1},\ldots,t^{i_s},t^{i_{s+1}};1)}\times$$

$$\times\int\limits_0^\infty\frac{\theta^{n-s}}{n!}e^{-\theta}.\frac{(d(t^{i_1},\ldots,t^{i_s},\frac{1}{\theta}x;1))^{n-s}}{(n-s)!},\cdot\prod_{k=1}^{s}\nu(t^{i_1},\ldots,t^{i_k},\frac{1}{\theta}\cdot x)d\theta.$$

The expression $\nu(t^{i_1},\ldots,t^{i_k},x/\theta)$ takes only two values 1 and 0. When $t^{i_1},\ldots,t^{i_k}$ and x are fixed, the values of $\theta>0$ for which $\nu(t^{i_1},\ldots,t^{i_k},x/\theta)=1$ coinside with solutions of the system of inequalities

$$\left|\begin{array}{c}\dfrac{d(t^{i_1},\ldots,t^{i_k},\frac{1}{\theta}x;1)}{d(t^{i_1},\ldots,t^{i_k};1)}\geq 0\\ \theta>0\end{array}\right. .$$

Applying the identity

$$d(t^{i_1},\ldots,t^{i_k},\frac{1}{\theta}x;1)=\frac{1}{\theta}(d(t^{i_1},\ldots,t^{i_k},x;0)+\theta d(t^{i_1},\ldots,t^{i_k},0^*;1))$$

to the above system, we get the following equivalent system

$$\left|\begin{array}{c}\theta>0\\ a(t^{i_1},\ldots,t^{i_j},x)\leq\theta\leq c(t^{i_1},\ldots,t^{i_j},x).\end{array}\right.$$

Consequently,

$$\left\{\theta:\prod_{k=1}^{s}\nu(t^{i_1},\ldots,t^{i_k},\frac{1}{\theta}x)=1\right\}=\bigcap_{k=1}^{5}\left\{\theta:\nu(t^{i_1},\ldots,t^{i_k},\frac{1}{\theta}x)=1\right\}=$$

$$=\bigcap_{k=1}^{s}\left\{\theta:\theta>0,a(t^{i_1},\ldots,t^{i_k},x)\leq\theta\leq c(t^{i_1},\ldots,t^{i_k},x)\right\}=$$

$$=\left\{\theta:A(t^{i_1},\ldots,t^{i_s},x)\leq\theta\leq B(t^{i_1},\ldots,t^{i_k},x)\right\}.$$

For the values of the following integral we have

$$\int\limits_{A(t^{i_1},\ldots,t^{i_s},x)}^{B(t^{i_1},\ldots,t^{i_s},x)}(\frac{(d(t^{i_1},\ldots,t^{i_s},x;0)+\theta d(t^{i_1},\ldots,t^{i_s},0^*;1))^{n-s}}{(n-s)!}e^{-\theta}d\theta=$$

$$= \sum_{j=0}^{n-s} \frac{(d(t^{i_1},\dots,t^{i_s},0^*;1))^j (d(t^{i_1},\dots,t^{i_s},x;0))^{n-s-j}}{(n-s-j)!} \times \int\limits_{A(t^{i_1},\dots,t^{i_s},x)}^{B(t^{i_1},\dots,t^{i_s},x)} \frac{\theta^j}{j!} e^{-\theta} d\theta.$$

After integration by parts in the last integral, we get the necessary expressions included in the density of η which completes the proof of the Theorem.

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ПЛЪТНОСТ НА ЛИНЕЙНИ ТРАНСФОРМАЦИИ ОТ НЕЗАВИСИМИ ЕКСПОНЕНЦИАЛНИ СЛУЧАЙНИ ВЕЛИЧИНИ

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Изведена е плътността на линейни трансформации от независими експоненциално разпределени случайни величини.