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NO-ARBITRAGE CONDITIONS FOR SYSTEMS OF FIXED EXCHANGE RATES

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This paper analyzes the issue of eliminating arbitrage opportunities in systems of countries with mutually fixed exchange rates. On the basis of graph-theoretic tools we suggest a method for calculating exchange rates that do not allow arbitrage. We also provide a means for checking whether a particular set of exchange rates gives rise to arbitrage opportunities.

1. Introduction. The present work tackles the problem of calculating a set of fixed exchange rates among a group of countries, with the values of the exchange rates having the property that no arbitrage opportunities arise in the system. This economic problem is formalized with the aid of graph-theoretic tools. The work suggests an approach for checking whether a set of fixed exchange rates admits arbitrage. It also offers a way to find sets of exchange rates for which there do not exist any arbitrage opportunities in the system.

2. Problem formulation. Take n countries, each of them having its own currency. Let us assume that these countries are preparing to join an integration area and each country has to fix its exchange rate vis-a-vis the currencies of the other countries. Is it possible to determine the exchange rates of the countries in such a way as to guarantee the absence of arbitrage opportunities in the system? In other words, can we pick such exchange rates that, starting with a unit of any currency and effecting an arbitrary chain of exchanges of one currency for another, we end up with exactly one unit of the original currency upon converting back to it? We note that the problem becomes nontrivial for the case of integration areas comprising more than three countries, whereas the cases of two and three countries are obvious.

One way to approach this problem is to formalize the system of countries and exchange rates as a graph whose vertices are identified with the countries, while each edge of the graph is assigned a positive[1] number representing the respective exchange rate. In order to have a well-defined system, for each number $\bar{x}$, representing the respective exchange rate, we also need to specify the direction of the exchange, i.e. are $\bar{x}$ units of country i's currency traded for one unit of country j's currency or, conversely, $\bar{x}$ units of country j's

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[1] Assigning a negative or zero number is meaningless from an economic viewpoint.

currency are traded for a unit of i's currency. In other words, the system is described by means of a directed graph whose edges have been assigned positive numbers. Then a sequence of exchanges can be viewed as describing a directed cycle in the graph and for the exchange rates that are part of the cycle we have (starting with one unit of the initial currency) $\bar{x}_{12}.\bar{x}_{23}.\bar{x}_{34}\ldots\bar{x}_{l1}$, where l is the number of elements in the cycle. The no-arbitrage condition for such a chain of exchanges is $\bar{x}_{12}.\bar{x}_{23}.\bar{x}_{34}\ldots\bar{x}_{l1} = 1$ and the no-arbitrage condition for the system as a whole is that the above is valid for any cycle in the graph.

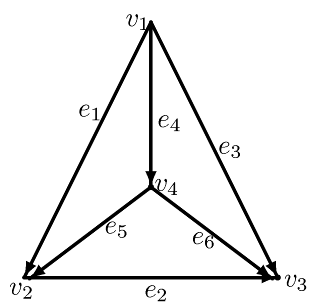
For analytical purposes it is more convenient to assign to each edge the logarithm of the respective exchange rate, denoted here by $x_{ij} = \ln\bar{x}_{ij}$, and not the value of the exchange rate itself. Then the numbers assigned to the edges can be real and the no-arbitrage condition will be $x_{12} + x_{23} + \ldots + x_{l1} = 0$ for any cycle in the graph (with the assumption that these rates as quoted here coincide with the orientation of the cycle).

3. Necessary graph-theoretic results. Let a finite directed graph $G = (V, \vec{E})$ be given, with V denoting the vertex set and $\vec{E}$ the edge set with the respective orientation for each of its elements, i.e. each edge has a direction and, respectively, an initial and terminal vertex. Formally, each edge can be defined through the ordered pair of the vertices it joins, with the initial vertex being first, i.e. $e_i = (v_l, v_k) \in \vec{E}$ for edge e_i with initial vertex v_l and terminal vertex v_k. We usually denote the cardinality of the vertex set by $|V| = n$ and the cardinality of the edge set by $|E| = m$. A graph G is called *complete* if any two vertices are joined by an edge. Here we shall be concerned only with complete directed graphs having exactly one edge between any two vertices[2]. It is clear from general combinatorial considerations that a complete graph with n vertices has $m = n(n-1)/2$ edges. A vertex v is called *incident* to edge e if e connects it to another vertex.

A directed graph G with vertices $\{v_1, \ldots, v_n\}$ and edges $\{e_1, \ldots, e_m\}$ can be represented through its incidence matrix $B = (b_{ij})_{n\times m}$, where

$$b_{ij} := \begin{cases} 1 & v_i \text{ is a terminal vertex for } e_j \\ -1 & v_i \text{ is an initial vertex for } e_j \\ 0 & \text{otherwise.} \end{cases}$$

An example of a complete directed graph and its incidence matrix is given in Fig. 1.



$$B = \begin{pmatrix} -1 & 0 & -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix}$$

Fig. 1. A complete directed graph with four vertices and its incidence matrix.

[2] Such graphs are sometimes called *tournaments*, see e.g. [1].

For a given directed graph a *directed path* from vertex v_1 to vertex v_l is defined as a finite alternating sequence of vertices and edges of the type $v_1e_1v_2e_2\ldots v_{l-1}e_{l-1}v_l$, such that the edges in the path have the same orientation, i.e. vertex v_i is a terminal vertex for edge e_{i-1} and an initial vertex for edge e_i. A reorientation of some of the edges in the sequence may be required in order to obtain a directed path. A *directed cycle* is a directed path for which only the first and the last vertex coincide. A directed cycle $\mathcal{C}$ in a directed graph may be described by means of a vector of elements taking values in the set $\{-1, 0, 1\}$. More precisely, such a vector $\mathbf{c}$ is an element of $\{-1, 0, 1\}^{|E|}$, where the edges of the graph have been labelled in advance and the jth component of $\mathbf{c}$ corresponds to the jth edge of the graph (see [2]). For a chosen orientation of the cycle $\mathcal{C}$ the corresponding vector $\mathbf{c}$ can be described componentwise according to the rule

$$c_j := \begin{cases} 1 & e_j \text{ is an element of } \mathcal{C} \text{ and their orientations coincide} \\ -1 & e_j \text{ is an element of } \mathcal{C} \text{ and their orientations do not coincide} \\ 0 & e_j \text{ is not an elemen t of } \mathcal{C}, \end{cases}$$

where c_j is the jth component of $\mathbf{c}$. If a vector $\mathbf{c}$ of the above type describes a cycle in a graph, its product with the incidence matrix of the graph produces a zero vector: $B\mathbf{c} = \mathbf{0}$.

The *cycle space* of a directed graph G is the linear space over the field of rational numbers $\mathbb{Q}$, spanned by the cycles of the graph as represented by their vectors. Each cycle of the graph is an element of this space. For connected graphs, such as the ones used in this work, the dimension of the cycle space is $k = |E| - |V| + 1 = m - n + 1$. The dimension of the cycle space is called the *cyclomatic number* of the graph.

A set of cycles in a directed graph G is called a *cycle basis* of G if it is a basis for the cycle space of the graph. Clearly, each cycle basis contains k cycles. Any cycle can be represented as a linear combination of the elements of the cycle basis. (The converse is not true.)

For our purposes it suffices to assume that a cycle basis is given and, respectively, we do not need to apply any algorithms for finding bases of directed cycles in a given graph. We note, however, that such bases can be obtained for example through directed spanning trees for the respective graph[3]. The cycle bases that can be obtained by means of directed spanning trees are called *strictly fundamental cycle bases* (see [2]). They have the property that any cycle in the graph can be obtained as an appropriate $\{0, \pm 1\}$ combination of elements of the basis.

4. Main results. Assume that the n countries under consideration have been mapped to the vertices of a complete directed graph, labelled respectively from 1 to n, and that the edges of the graph have been numbered from 1 to m as in the previous section. Let us assign to edge i a number x_i corresponding to the logarithm of the respective exchange rate for direction of the exchange coinciding with the direction of the edge. For brevity we write $\mathbf{x} := (x_1, \ldots, x_m)'$.

If a cycle basis $\mathbf{c}_1, \ldots, \mathbf{c}_k$ is given for a directed graph, the no-arbitrage condition for the ith cycle of the basis is

$$\mathbf{x}'\mathbf{c}_i = 0. \tag{1}$$

If the no-arbitrage condition (1) is valid for any cycle in the basis, we write in matrix

[3] For the methods of construction of directed spanning trees see e.g. [4].

terms

$$\mathbf{x}'\mathbf{C} = (0, \ldots, 0), \tag{2}$$

where $\mathbf{C} := [\mathbf{c}_1, \ldots, \mathbf{c}_k]$. Then we have the following

Proposition 1. *If the no-arbitrage condition is valid for the elements of the cycle basis, it is also valid for any cycle in the graph.*

Indeed, for an arbitrary cycle $\mathbf{c}$ we have

$$\mathbf{x}'\mathbf{c} = \mathbf{x}'\left(\sum_{i=1}^{k} \lambda_i \mathbf{c}_i\right) = \sum_{i=1}^{k} \lambda_i \mathbf{x}'\mathbf{c}_i = 0, \tag{3}$$

where $\lambda_i \in \mathbb{Q},\ i = 1, \ldots, k$ are the coefficients in the linear combination used to represent the cycle $\mathbf{c}$ in the given basis.

Let the no-arbitrage condition be valid for the elements of a given cycle basis. Then it is valid for any other cycle basis in the graph. The last claim follows from the fact that the element $\tilde{\mathbf{c}}_j$ of the new cycle basis has the representation

$$\tilde{\mathbf{c}}_j = \sum_{i=1}^{k} \lambda_{ij} \mathbf{c}_i \quad \Rightarrow \quad \mathbf{x}'\tilde{\mathbf{c}}_j = \mathbf{x}'\left(\sum_{i=1}^{k} \lambda_{ij} \mathbf{c}_i\right) = \sum_{i=1}^{k} \lambda_{ij} \mathbf{x}'\mathbf{c}_i = 0.$$

Consequently, the problem of finding a set of exchange rates for which there are no arbitrage opportunities in the system reduces to the problem of finding a vector $\mathbf{x}$ which satisfies the no-arbitrage condition for an arbitrary cycle basis.

Let us take the cycle basis $\mathbf{c}_1, \ldots, \mathbf{c}_k$ represented in matrix terms with the $\mathbf{C}$ as above. The vector $\mathbf{x}$ is obtained as a solution to the homogeneous system

$$\mathbf{C}'\mathbf{x} = \mathbf{0}. \tag{4}$$

Obviously the system always has the trivial zero solution. The existence of a nontrivial solution depends on the rank of the matrix $\mathbf{C}$, which in our case is k since the columns of the matrix are the elements of the cycle basis. There exist nontrivial solutions if the number of free variables in the system (the difference between the number of unknowns and the rank of the matrix of the system) is positive [3, Chapter 2]. In our case we have $m - k = n(n-1)/2 - n(n-1)/2 + n - 1 = n - 1 > 0$. In summary, we have

Proposition 2. *The vector* $\mathbf{x}$ *is a solution of* (4). *Such systems always possess nontrivial solutions, with the fundamental system of solutions having* $m - k = n - 1$ *elements.*

Equation (4) allows us to check whether a set of exchange rates creates arbitrage opportunities in the system:

Corollary 1. *Arbitrage opportunities exist if the vector of the logarithms of the exchange rates* $\mathbf{x}$ *does not satisfy.*

Conversely, in solving the problem of finding a set of exchange rates that do not admit arbitrage, their values are obtained after taking the antilog of the elements of the solution $\mathbf{x}$.

The fact that algebraic systems of the above type have free variables is of substantial importance for decisions about the levels of the fixed exchange rates within an integration area. It means that decision-makers have enough freedom to reconcile the choice of exchange rates with other factors that matter for the conduct of economic policy. As an example, when fixing the exchange rates this freedom may be used to ensure (arbitrage-free) adjustment of the price levels in the countries participating in the integration area.

More generally, if we define appropriate objective functions and constraints on the set of admissible solutions $\mathbf{x}$, such situations can be formalized as optimization problems whose solutions are arbitrage-free systems of exchange rates that are optimal with respect to the chosen criterion.

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УСЛОВИЯ ЗА ОТСЪСТВИЕ НА АРБИТРАЖ ЗА СИСТЕМИ ОТ ФИКСИРАНИ ВАЛУТНИ КУРСОВЕ

Андрей А. Василев

Статията анализира въпроса за елиминиране на възможности за арбитраж в системи от страни с валутни курсове, които са фиксирани едни спрямо други. На основата на инструментариум от теория на графите е предложен метод за изчисляване на валутните курсове, които не позволяват арбитраж. Даден е начин да се проверява дали конкретен набор от валутни курсове оставя възможности да бъде осъществен арбитраж.