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TO PLAY WITH ε AND δ

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In this note we consider several problems involving definitions of some basic mathematical concepts using the so called (ε,δ)-language. The results can be used by lecturers in mathematics.

Introduction. The language of (ε,δ) (or, briefly, LED) is a common tool in defining basic concepts in Mathematical Analysis such as limits and continuity. In this note we consider some issues in the definition of continuity by using LED and their possible application in teaching of mathematics. We also consider some not very popular concepts of limits and derivatives for functions defined on countable or even finite sets.

The definition of continuity. In this section we deal with the standard definition of continuity using the language of (ε,δ). For simplicity we shall consider functions $f : T \to \mathbb{R}$, where the set $T \subset \mathbb{R}$ is non-empty (but may not be an interval). To avoid trivial results we can further assume that T contains at least two points. Also, for convenience of the reader, we recall the definitions of continuity.

Definition 1. *The function f is said to be:* 1) *continuous at the point $x_0 \in T$ if for any $\varepsilon > 0$ there exists $\delta = \delta(x_0,\varepsilon) > 0$ such that for all $x \in T$ the inequality $|x - x_0| \leq \delta$ implies $|f(x) - f(x_0)| \leq \varepsilon$, and continuous on T if it is continuous for all $t \in T$;* 2) *uniformly continuous on T if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x, x_0 \in T$ the inequality $|x - x_0| \leq \delta$ implies $|f(x) - f(x_0)| \leq \varepsilon$.*

Let us "slightly" change the definition of uniform continuity, see also [1].

Let F_1, F_2 and F_3 be the sets of functions f such that, respectively: **(i)** for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that for all $x, x_0 \in T$ the inequality $|x - x_0| \leq \delta$ implies $|f(x) - f(x_0)| \leq \varepsilon$; **(ii)** for any ε there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x, x_0 \in T$ the inequality $|x - x_0| \leq \delta$ implies $|f(x) - f(x_0)| \leq \varepsilon$; **(iii)** for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x, x_0 \in T$ the inequality $|f(x) - x_0| > \delta$ implies $|x - f(x_0)| \leq \varepsilon$.

In (i) we have "forgotten" the inequality $\delta > 0$, while in (ii) we missed the inequality $\varepsilon > 0$ – such errors may often be found in a student exam paper. In (iii) it seems like someone has exchanged $f(x)$ and x and replaced $\leq \delta$ by $> \delta$. The result of these changes is described below.

(i) The set F_1 consists of all functions. (ii) The set F_2 is empty. (iii) If the interval T is finite, then F_3 is the set of bounded functions.

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To prove (i) for $\varepsilon > 0$ choose $\delta = -1$. Then inequality $|x - x_0| \le \delta = -1$ is never fulfilled and implies anything, including $|f(x) - f(x_0)| \le \varepsilon$.

(ii) For $f \in F_2$ and $\varepsilon = -1$ choose the corresponding $\delta > 0$. Then for $x = x_0$ we have $0 = |x - x| \le \delta$ and this implies $0 = |f(x) - f(x)| \le -1$ which is impossible.

(iii) Let B be the set of bounded functions $T \to \mathbb{R}$ and denote $m := \sup\{|x| : x \in T\}$. For $f \in F_3$ and $\varepsilon > 0$ choose the corresponding $\delta > 0$. We shall show that $|f(x)| \le m + \varepsilon + \delta$ for all $x \in T$. Indeed, suppose the opposite, i.e. that there is $x_0 \in T$ such that $|f(x_0)| > m + \varepsilon + \delta$. Then, taking $x = x_0$, we get

$$|f(x) - x_0| = |f(x) - x| \ge |f(x)| - |x| > m + \varepsilon + \delta - m = \varepsilon + \delta > \delta.$$

At the same time we also have $|x - f(x_0)| = |x - f(x)| > \varepsilon + \delta > \varepsilon$, which is a contradiction to the definition of F_3. Hence f is bounded, i.e. $F_3 \subset B$.

Conversely, suppose that f is bounded, i.e. $|f(x)| \le M < \infty$ for all $x \in T$. For any $\varepsilon > 0$ take $\delta := M + m$. Then $|f(x) - x_0| \le |f(x)| + |x_0| \le M + m = \delta$. Hence the iniquality $|f(x) - x_0| > \delta$ cannot be satisfied and implies whatever, including $|x - f(x_0)| \le \varepsilon$. Therefore $B \subset F_3$ and we have proved (iii).

Consider now the following definition.

The function $f : T \to \mathbb{R}$ is uniformly continuous on T if for any $\varepsilon > 0$ there is a quantity $\delta = \delta(\varepsilon) > 0$ such that for all $x, x_0 \in T$ the inequality $|x - x_0| \le \delta$ implies $f(x) - f(x_0) \le \varepsilon$.

This definition seems to be faked compared with proposition 2) in Definition 1 since we have the inequality $f(x) - f(x_0) \le \varepsilon$ instead of $|f(x) - f(x_0)| \le \varepsilon$. But things are in fact OK: it suffices to change the arguments x and x_0, satisfying $|x - x_0| \le \delta$. Then $f(x) - f(x_0) \le \varepsilon$ and $f(x_0) - f(x) \le \varepsilon$ which is equivalent to $|f(x) - f(x_0)| \le \varepsilon$.

Consider finally one more set defined in terms of LED.

Let F_4 be the set of functions f such that for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x, x_0 \in T$ the inequality $|f(x) - f(x_0)| \le \delta$ implies $|x - x_0| \le \varepsilon$.

Then the set F_4 consists of functions having uniformly continuous inverse.

For $f \in F_4$ the inequality $x \ne x_0$ implies $f(x) \ne f(x_0)$. Indeed, suppose the opposite, i.e. that $f(x) = f(x_0)$. Take $\varepsilon := |x - x_0|/2$. Then $0 = |f(x) - f(x_0)| \le \delta(\varepsilon)$ implies $|x - x_0| \le \varepsilon = |x - x_0|/2$ and $x = x_0$ which is a contradiction. Hence the function $g := f^{-1} : f(T) \to T$ exists. Setting $y := f(x)$, $y_0 := f(x_0)$ we see that for $y, y_0 \in f(T)$ the inequality $|y - y_0| \le \delta$ implies $|x - x_0| = |g(y) - g(y_0)| \le \varepsilon$ which means that g is uniformly continuous.

Consider now a function $f : T \to \mathbb{R}$ which has a uniformly continuous inverse $g : f(T) \to T$. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for $y, y_0 \in f(T)$ the inequality $|y - y_0| = |f(x) - f(x_0)| \le \delta$ implies $|g(y) - g(y_0)| = |x - x_0| \le \varepsilon$.

More examples, which are actually plays with (ε, δ)-definitions, can be found in [1].

Strange limits and derivatives. In this section we assume that X, Y are Banach spaces over $\mathbb{R}$ or $\mathbb{C}$, the set $U \subset X$ contains at least two points, and $f : U \to Y$ is a given function.

As a rule, limits for f are defined when U has non-empty interior and in particular when U is open. Furthermore, when a limit of f at $x_0 \in X$ is defined, then usually only values $f(x)$ for $x \ne x_0$ are used, i.e. the argument x is prevented from reaching x_0. And while this is natural for $x_0 \notin U$, it is still possible to allow $x = x_0$ when $x_0 \in U$.

Generally, it is accepted that the case $x = x_0$ must be excluded for being "noninteresting". Wheather this is really the case is, however, not very clear, see e.g. [2]. Anyway, from purely didactical reasons, one can consider the possibility $x = x_0$ as an exercise in the course of Mathematical Analysis.

In order to deal with different concepts of limits, we need the next definition.

Definition 2. *The point $x_0 \in X$ is an* accumulation point *for $U \subset X$ if for any $\varepsilon > 0$ there exists $x \in U$ such that $0 < \|x - x_0\| \le \varepsilon$. Further on, $x_0 \in U$ is an* isolated point *of U if it is not an accumulation point for U. An accumulation point $x_0 \in X\backslash U$ is a singular point for the function $f : U \to Y$.*

Denote by $\overline{U}$ the closure of U. Each $x \in \overline{U}$ is either an element of U or a singular point for f. Next we recall the definition of a limit using LED.

Definition 3. *The function f has a limit $y_0 \in Y$ at the accumulation point $x_0 \in X$ for U if for any $\varepsilon > 0$ there exists $\delta = \delta(x_0, \varepsilon) > 0$ such that for all $x \in U$ the inequality $0 < \|x - x_0\| \le \delta$ implies $\|f(x) - y_0\| \le \varepsilon$.*

There are three strict inequalities in Definition 3, namely $0 < \|x - x_0\|$, $\delta > 0$ and $\varepsilon > 0$, and we shall see what happens when we delete them one by one thus obtaining "strange" limits.

The requirement that x_0 is accumulation point for U is connected with the inequality $0 < \|x - x_0\|$. Indeed, if $x_0 \in U$ is an isolated point of U, then in Definition 3 one may choose $\delta = d/2$, where $d > 0$ is the distance between x_0 and the set $U\backslash\{x_0\}$. Then relations $x \in U$ and $0 < \|x - x_0\| \le d/2$ are contradictory and imply everything, i.e. any $y \in Y$ should be considered as limit of f at x_0. Also, if the function f has a limit at certain point, then the set U is at least countable.

Definition 4. *The function f has pseudolimit I $y_1 \in Y$ at the point $x_0 \in \overline{U}$ if for any $\varepsilon > 0$ there exists $\delta = \delta(x_0, \varepsilon) > 0$ such that for all $x \in U$ the inequality $\|x - x_0\| \le \delta$ implies $\|f(x) - y_1\| \le \varepsilon$.*

Definition 4 differs from Definition 3 by deleting the reqirement that x_0 is accumulation point for U, and by allowing x to become x_0. Thus the possibility to take the limit $x \to x_0$ is not preassumed.

Definition 5. *The function f has pseudolimit II $y_2 \in Y$ at the point $x_0 \in \overline{U}$ if for any $\varepsilon > 0$ there exists $\delta = \delta(x_0, \varepsilon) \ge 0$ such that for all $x \in U$ the inequality $\|x - x_0\| \le \delta$ implies $\|f(x) - y_2\| \le \varepsilon$.*

Here we go one step further in comparison with Definition 4 allowing the quantity also δ to become zero.

Definition 5 is very narrow since for each $x_0 \in U$ the function f has pseudolimit II $f(x_0)$. Moreover, if $x_0 \in \overline{U}\backslash U$ is a singular point of f, then any y_2 is a pseudolimit II of f at x_0. Indeed, for $\varepsilon > 0$ choose $\delta = 0$. Then relations $x \in U$ and $\|x - x_0\| = 0$ give $x = x_0 \in U$ which is impossible since $x_0 \notin U$. Hence the inequality $\|f(x) - y_2\| \le \varepsilon$ has not to be checked.

Replacing the inequality $\varepsilon > 0$ by $\varepsilon \ge 0$ in Definition 5, we come to the definition of pseudolimit III of f at x_0. It can be shown that pseudolimits II and III coincide.

It seems reasonable to relax the inequality $\delta \ge 0$ when $x \notin U$. Thus we come to the next concept.

Definition 6. *The function f has pseudolimit II-a $y_2 \in Y$ at the point $x_0 \in \overline{U}$ if for any $\varepsilon > 0$ there exists $\delta = \delta(x_0, \varepsilon) \geq 0$, satisfying $\delta(x_0, \varepsilon) > 0$ for $x_0 \in \overline{U} \backslash U$, and such that for all $x \in U$ the inequality $\|x - x_0\| \leq \delta$ implies $\|f(x) - y_2\| \leq \varepsilon$.*

When $x_0 \in U$ the pseudolimit II-a of f at x_0 is $f(x_0)$. If $x_0 \in \overline{U} \backslash U$ then pseudolimit II-a and pseudolimit I are equivalent.

Replacing the inequality $\varepsilon > 0$ by $\varepsilon \geq 0$ in Definition 6, we can define pseudolimit III-a $y_3 \in Y$ of f a x_0.

For $x_0 \in U$ the pseudolimit III-a of f at x_0 is $f(x_0)$. If, however, $x_0 \notin U$, then we can take $\varepsilon = 0$. Then $f(x) = y_3$ for all $x \in U$ satisfying $0 < \|x - x_0\| \leq \delta$.

Now it is clear that if a limit and a pseudolimit simultaneously exist at certain point, they may be different.

It is instructive to draw the attention of the students to the following facts, summarized in a theorem.

Theorem 1. *Let $x_0 \in U$. Then the following assertions take place: 1) if f has limit y_0 at x_0 and $y_0 = f(x_0)$ then f is continuous at x_0; 2) if f has pseudolimit I y_1 at x_0 then $y_1 = f(x_0)$ and f is continuous at x_0; 3) the function f has pseudolimits II, II-a and III-a, and all they are equal to $f(x_0)$.*

Statement 1 is the definition of continuity at an accumulation point. However, one has to postulate that f is continuous at isolated points. To prove 2 recall that according to Definition 4, for any $\varepsilon > 0$ there is $\delta > 0$, such that for $x \in U$ the inequality $\|x - x_0\| \leq \delta$ implies $\|f(x) - y_1\| \leq \varepsilon$. Taking $x = x_0$ we get $\|f(x_0) - y_1\| < \varepsilon$ and since $\varepsilon > 0$ is arbitrary, it follows that $f(x_0) = y_1$. Statement 3 follows directly from the definitions.

Next we shall consider the interconnection between limits and pseudolimits.

It follows from the definitions that a function f has no limit at an isolated point $x_0 \in U$, but has pseudolimits I, II, II-a and III-a, all equal to $f(x_0)$, at such a point. So it remains to analyze the case when $x_0 \in \overline{U}$ is an accumulation point for U.

Let us first suppose that $x_0 \in X \backslash U$. If f has a limit y_0 at x_0, then it has also pseudolimit I y_0 at x_0. Indeed, here we should assume $x \neq x_0$ and $\|x - x_0\| > 0$. But then we come to the definition of a limit. At the same time any element $y_2 \in Y$ is pseudolimit II of f at x_0.

Suppose now that $x_0 \in U$, i.e. that the quantity $f(x_0)$ is defined. We already know that f has pseudolimits II, II-a and III-a $f(x_0)$ at x_0.

If f has a limit y_0 at x_0, then we have two possibilities. First, if $y_0 = f(x_0)$ (i.e. if f is continuous at x_0), then f has also pseudolimit I at x_0, equal $f(x_0)$. Second, if $y_0 \neq f(x_0)$ (i.e. if f has a removable discontinuity at x_0), then the function f cannot have pseudolimit I at x_0 because it is not continuous at this point.

Let us now assume that f has no limit at the accumulation point $x_0 \in U$ for U but has there pseudolimit I y_1. The lack of limit means that for some $\varepsilon > 0$ and all $\delta > 0$ there exists $x \in U$, such that $0 < \|x - x_0\| < \delta$ and $\|f(x) - y_1\| \geq \varepsilon$. But this is a contradiction to the assumption that y_1 is pseudolimit I of f at x_0. Hence pseudolimit I of f at x_0 does not exist.

The above results can be summirazied below.

Theorem 2. *The following assertions take place.*

- *Let $x_0 \in U$ be an isolated point of U. Then f has no limit but has pseudolimits I, II, II-a and III-a at x_0, all equal to $f(x_0)$.*

- *Let $x_0 \in U$ be an accumulation point for U. Then:*

  – f has pseudolimits II, II-a and III-a $f(x_0)$ at x_0;

  – if f has limit $f(x_0)$ at x_0 (i.e. if f is continuous at x_0), then it also has pseudolimit I $f(x_0)$ at x_0;

  – if f has limit $y_0 \neq f(x_0)$ at x_0 (i.e. if f has a removable discontinuity at x_0) then pseudolimit I at x_0 does not exist;

  – if f has pseudolimit I at x_0 then it is equal to $f(x_0)$, and f is even continuous at x_0 (and the limit $f(x_0)$ of f at x_0 exists as a corollary).

- *Let $x_0 \in \overline{U}\backslash U$ be an accumulation point for U Then:*

  – the limit, pseudolimit I and pseudolimit II-a are equivalent (i.e. they exist simultaneously at x_0 and are equal, or no one of them exists);

  – each $y_2 \in Y$ is pseudolimit II of f at x_0.

  – pseudolimit III-a exists at x_0 if and only if there is $\delta > 0$ such that $f(x)$ is identically constant for $0 < \|x - x_0\| \le \delta$.

Next we shall briefly discuss "strange" derivatives for functions, defined on countable or even finite sets. Although they are not strange from, say, a constructive viewpoint. Moreover, in practical computations we actually work with rational numbers only and hence – with strange derivatives.

Suppose that $X = \mathbb{R}$ and the set $U \subset \mathbb{R}$ contains at least two points. Consider the function $F : U_0 \to Y$, defined by $F(x) := (f(x) - f(x_0))/(x - x_0)$, $x \in U_0 := U\backslash\{x_0\}$.

The standard derivative $f'(x_0) \in Y$ (we identify the space of bounded linear operators $\mathbb{R} \to Y$ with Y) is the limit of F at x_0 whenever this limit exists. In this case $x_0 \in U$ is an accumulation point of U and a singular point for F. As we already know, in this case the limit, pseudolimit I and pseudolimit II-a of F at x_0 are equivalent.

Usually, when dealing with derivatives it is supposed that the set U is an interval or a sum of intervals. In fact, as the above definition suggests, the only requirement on U follows from the existence of a limit: U must have an accumulation point and, hence, must be at least countable. Examples for such sets are $U = \mathbb{Q}$ (the set of rationals), $U := \{1/n : n \in \mathbb{N}\} \cup \{0\}$ or $U := \{1/p : p \in \mathbb{P}\} \cup \{0\}$, where $\mathbb{P}$ is the set ot primes.

In computational practice one has to work with finite sets of machine numbers. Hence it makes sense to define derivatives at isolated points of U.

Let $x_0 \in U$ be an isolated point of the set U containing at least two points. Then F is well defined on $U_0 = U\backslash\{x_0\}$. Denote also $f_0 := f(x_0)$. In defining the derivative at x_0 we shall consider three cases.

First, suppose that $x_0 = \inf\{U\}$. Denote $x_2 := \inf\{U_0\} > x_0$. If $x_2 \in U$ define the derivative of f at x_0 as $f'(x_0) := (f(x_2) - f_0)/(x_2 - x_0)$. If $x_2 \notin U$ but f has limit f_2 at x_2, set $f'(x_0) := (f_2 - f_0)/(x_2 - x_0)$. Finally, if $x_2 \notin U$ and f has no limit at x_2, the derivative of f at x_0 is not defined.

Second, let $x_0 = \sup\{U\}$ and set $x_1 := \sup\{U_0\} < x_0$. We have $x_0 > x_1$. If $x_1 \in U$ the derivative of f at x_0 is defined as $f'(x_0) := (f(x_1) - f_0)/(x_1 - x_0)$. If $x_1 \notin U$ but f has limit f_1 at x_1, set $f'(x_0) := (f_1 - f_0)/(x_1 - x_0)$. Finally, if $x_1 \notin U$ and f has no limit at x_2, then f has no derivative at x_0.

Third, suppose that $x_1 := \sup\{x \in U, x < x_0\} < x_0 < x_2 := \inf\{x \in U : x > x_0\}$. If $x_1 \in U$ define the left derivative of f at x_0 as $f'(x_0 - 0) := (f(x_1) - f_0)/(x_1 - x_0)$. If $x_1 \notin U$ but f has limit f_1 at x_1, set $f'(x_0 - 0) := (f_1 - f_0)/(x_1 - x_0)$. Finally, if $x_1 \notin U$ and f has no limit at x_1, the left derivative of f at x_0 is not defined. The right derivative $f'(x_0+0)$ of f at x_0 is defined similarly. We can define also a derivative $f'(x_0)$ as $f'(x_0-0) = f'(x_0+0)$ whenever the latter equality holds. But this definition of $f'(x_0)$ is a restricted concept. Let e.g. $U = \{x_1, x_0, x_2\}$, $x_1 < x_0 < x_2$, and denote $f_k = f(x_k)$. Then $f'(x_0)$ exists if and only if $(f_1 - f_0)/(x_1 - x_0) = (f_2 - f_0)/(x_2 - x_0)$, which yields $f(x) = ax + b$, $x \in U$.

Generalizations. The results are easily generalized to the case of (complete) metric spaces. A further generalization to topological spaces, however, is not straightforward. Indeed, the play with ε and δ in the form presented above requires some type of metrics. An attempt to extend some of the results about uniform continuity may eventually involve the concept of uniform topological spaces.

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ИГРИ С ε И δ

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В тази бележка са разгледани някои задачи, свързани с дефиниране на основни математически понятия с използване на езика (ε, δ). Резултатите могат да се използват от преподавателите по математика.