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USING CALCULUS FOR SOLVING CUBIC EQUATIONS

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The problem of solving cubic equations is not among the favorite subjects in teaching at the standard high schools in Bulgaria. Probably the reason is that in the "algebraic" method of view the procedure of solving the cubic equations requires some knowledge about what is a complex number.
In this note we make an attempt to offer a "non algebraic" point of view for solving cubic equations using just the typical humble abilities of our middle school educational system at present.

1. Basic algebra. We shall deal in this paper with solving the cubic equation

(ce) $$Ay^3 + By^2 + Cy + D = 0, \quad \text{where} \quad A \neq 0.$$

As it is well known, under the substitution $y = x - \dfrac{B}{3A}$, the equation **(ce)** takes the form

$$Ax^3 + C^*x + D^* = 0.$$

In this paper we shall consider the following special reduced forms of **(ce)**:

($\mathbf{r}_+$) $$4x^3 + 3p^2x + q^3 = 0$$

and

($\mathbf{r}_-$) $$4x^3 - 3p^2x + q^3 = 0$$

which are the result of dividing $Ax^3 + C^*x + D^* = 0$ by $A/4$.

As we shall see below, there is a considerable difference between cases ($\mathbf{r}_+$) (reduced form) and ($\mathbf{r}_-$) (casus irreducibilis).

Clearly, in every case the equation **(ce)** can be reduced to one of the equations ($\mathbf{r}_+$) or ($\mathbf{r}_-$). Hence, in what follows we shall deal with the functions $f_\pm(x) = 4x^3 \pm 3p^2x + q^3$ with $p > 0$.

2. Hyperbolic functions. Here we consider hyperbolic functions with a base $a > 0$; $a \neq 1$. For definiteness, we suppose that $a > 1$. Note, that it is not difficult to present in the middle school the hyperbolic function with some known base (for example $a = 2$).

2.1. And so, the *hyperbolic sine* is the function $\mathrm{sh}\,(u) = \frac{1}{2}(a^u - a^{-u})$. From the definition it follows that sh is defined for all values of u. The function has one zero at $u = 0$. If $u \to +\infty$, then a^{-u} becomes arbitrarily small (remember that $a > 1$). Because at the same time a^u increases without limit, the function become arbitrarily large. As $u \to -\infty$, a^{-u} becomes arbitrarily large and a^u approaches the value zero, so that the value of the function tends to $-\infty$.

Further, from the defining equation it follows that $\mathrm{sh}\,u = -\mathrm{sh}\,(-u)$. Thus, the function is *odd* and its graph is *centrally symmetric* with respect to the origin of coordinates; the range is $-\infty < u < \infty$.

2.2. The *hyperbolic cosine* is also defined for all values of u by the equation $\mathrm{ch}\,u = \frac{1}{2}(a^u + a^{-u})$. The function is *even* and its graph is *symmetrical with respect to the second axis.* Note, that $\mathrm{ch}\,u \geq 1$ for all values of u because $\alpha + 1/\alpha \geq 2$ whenever $\alpha > 0$. The equality occurs only for $\alpha = 1$ and hence $\mathrm{ch}\,u = 1$ only for $u = 0$.

2.3. *The functions* $\mathrm{th}\,u = \mathrm{sh}\,u/\mathrm{ch}\,u$ *and* $\mathrm{cth}\,u = \mathrm{ch}\,u/\mathrm{sh}\,u$. The first of these two functions (the hyperbolic tangent) is defined for all values of u, but for the second (the hyperbolic cotangent) the value $u = 0$ must be excluded. The range of $\mathrm{th}\,u$ is bounded: $-1 < \mathrm{th}\,u < 1$. By contrast, the range of $\mathrm{cth}\,u$ coincides with the sum $(-\infty, -1) \cup (1, \infty)$. Clearly both functions are *odd*.

2.4. Algebraic relations between the hyperbolic functions. From the equations of the functions the following well known identities are immediate consequences:

(hf) $$\mathrm{sh}\,(u+v) = \mathrm{sh}\,u\mathrm{ch}\,v + \mathrm{ch}\,u\mathrm{sh}\,v \quad \text{and} \quad \mathrm{ch}\,(u+v) = \mathrm{ch}\,u\mathrm{ch}\,v + \mathrm{sh}\,u\mathrm{sh}\,v.$$

Executing **(sh)**, it is easy to calculate that:

($\mathbf{hr}_+$) $$\mathrm{sh}\,3u = 4\mathrm{sh}^3 u + 3\mathrm{sh}\,u$$

and

($\mathbf{hr}_-$) $$\mathrm{ch}\,3u = 4\mathrm{ch}^3 u - 3\mathrm{ch}\,u.$$

2.5. One may add the identity $\mathrm{ch}^2 u - \mathrm{sh}^2 u = 1$ to realize the close similarity of these relations to those between the trigonometric functions. This justifies the use of the terms hyperbolic sine, cosine and so on. One important difference between the two classes consist in the fact, that it is easy to find out the inverse functions of the hyperbolic functions. Because of that, it is possible (and useful in our opinion) to teach hyperbolic functions in the secondary schools.

2.6. As a good exercise, it is worth checking the correctness of the following identities:

($\mathbf{t_3}$) $$\mathrm{th}\,3u = \frac{\mathrm{th}^3 u + 3\mathrm{th}\,u}{1 + 3\mathrm{th}\,u} \quad \text{and} \quad \mathrm{cth}\,3u = \frac{\mathrm{cth}^3 u + 3\mathrm{cth}\,u}{1 + 3\mathrm{cth}\,u}.$$

3. The inverse functions of the hyperbolic functions. The hyperbolic functions $\mathrm{sh}\,u$ and $\mathrm{th}\,u$ are invertible since they are monotonic. The function $\mathrm{cth}\,u$ is invertible since it is one-to-one. For $\mathrm{ch}\,u$ an inverse function can be defined in each of the two intervals $(-\infty, 0]$ and $[0, \infty)$ in which the function is monotonic.

Each of the above mentioned facts can be established by using appropriate tools from the secondary school. For example, the function $\mathrm{ch}\,u$ is decreasing in the interval $(-\infty, 0]$ because for the function $g(t) = t + 1/t$ we have $g'(t) = 1 - 1/t^2 < 0$ for every $t \in (0, 1]$.

But it is not necessary to investigate the above functions because we are going to find out their inverse values in the following text.

3.1. Inverse hyperbolic sine $u = \mathrm{sh}^{-1} v$. We can find this function by solving the equation $v = \frac{1}{2}(a^u - a^{-u})$ for given v. Multiplying by $2a^u$ gives first $2va^u = a^{2u} - 1$, or

$a^{2u}-2va^u-1=0$. This is a quadratic equation for a^u. Only the solution $a^u=v+\sqrt{v^2+1}$ is relevant, because $v-\sqrt{v^2+1}$ is always negative, whereas a^u can take only positive values.

Taking logarithms finally yields

(is) $$u=\log_a\left(v+\sqrt{v^2+1}\right),$$

which gives the *explicit form of the equation of the inverse function.*

3.2. Inverse hyperbolic cosine $u=\operatorname{ch}^{-1}v$. To invert $v=\operatorname{ch}u$ one proceeds as for $v=\operatorname{sh}u$ by similar steps to the equation $a^{2u}-2va^u+1=0$, leading to $a^u=v\pm\sqrt{v^2-1}$. Finally one obtains the inverse functions

(**ic**$_-$) $$u=\log_a\left(v-\sqrt{v^2-1}\right);\ \text{for the interval } (-\infty,0]$$

and

(**ic**$_+$) $$u=\log_a\left(v+\sqrt{v^2-1}\right);\ \text{for the interval } [0,+\infty).$$

3.3. Problem. Prove in a similar way that

(**it**) $$u=\operatorname{th}^{-1}(v)=\frac{1}{2}\log_a\left(\frac{1+v}{1-v}\right);v\in(-1,1)$$

and

(**ict**) $$u=\operatorname{cth}^{-1}(v)=\frac{1}{2}\log_a\left(\frac{1+v}{v-1}\right);v\in(-\infty,-1)\cup(1,\infty).$$

4. The cubic equations. Next we are going to apply the above considerations for solving cubic equations.

4.1. The case (r_+). Here we have the equation $f_+=4x^3+3p^2x+q^3=0$. For the derivative of f'_+ we have $f'_+(X)=3(4x^2+p^2)>0$ for all values of x. Thus the function f_+ is strictly increasing and the equation (**r**$_+$) has at most one root.

To find it, we make the substitution $x=p\operatorname{sh}u$. According to (**hr**$_+$), the equation (**r**$_+$) takes the form $p^3\operatorname{sh}3u+q^3=0$. Hence $\operatorname{sh}3u=-\dfrac{q^3}{p^3}$. Next, one can use (**is**) to obtain u:

$$3u=\log_a\left(-\frac{q^3}{p^3}+\sqrt{\frac{q^6}{p^6}+1}\right);\ \text{therefore } u=\frac{1}{3}\log_a\frac{\sqrt{p^6+q^6}-q^3}{p^3}$$

and finally we have

$$u=\log_a\frac{\sqrt[3]{\sqrt{p^6+q^6}-q^3}}{p}=\log_a\lambda,$$

where $\lambda=\dfrac{\sqrt[3]{\sqrt{p^6+q^6}-q^3}}{p}$. Now we can replace u in the equality $x=p\operatorname{sh}u$. We have $x=\dfrac{p}{2}(a^u-a^{-u})$ and because of the identity $a^u=a^{\log_a\lambda}=\lambda$ we obtain the root x:

$x = \frac{p}{2}\left(\lambda - \frac{1}{\lambda}\right)$. Hence

$$x=\frac{p}{2}\left(\frac{\sqrt[3]{\sqrt{p^6+q^6}-q^3}}{p}-\frac{p}{\sqrt[3]{\sqrt{p^6+q^6}-q^3}}\right)=\frac{1}{2}\left(\sqrt[3]{\sqrt{p^6+q^6}-q^3}-\sqrt[3]{\sqrt{p^6+q^6}+q^3}\right)$$

4.2. Example. We should note, that the above formula gives complicated expressions for the roots. Let us solve for example the equation $x^3+x-2=0$. Clearly we have to multiply it by 4 to obtain the reduced form $(\mathbf{r}_+)$. Then $p=\frac{2}{\sqrt{3}}$ and $q=-2$. Next, one may apply the above considerations to obtain that $\lambda=\sqrt[3]{\sqrt{28}+\sqrt{27}}$; therefore $\frac{1}{\lambda}=\sqrt[3]{\sqrt{28}-\sqrt{27}}$. Then the only root x_0 of our equation is

$$x_0=\frac{1}{\sqrt{3}}\left(\sqrt[3]{\sqrt{28}+\sqrt{27}}-\sqrt[3]{\sqrt{28}-\sqrt{27}}\right).$$

In the same time it is obvious that $x_0=1$ is a root of our equation. Note, that the Cardano's formula gives (naturally) very similar result: $x_0=\sqrt[3]{1+\sqrt{\frac{28}{27}}}+\sqrt[3]{1-\sqrt{\frac{28}{27}}}$.

Note however that the formula $(\mathbf{s}_+)$ gives the solution in every case. We offer to the reader as an exercise to solve the equations $x^3+x+2=0$, $x^3+x+1=0$ and $4x^3+3x+a=0$.

4.3. The case $(\boldsymbol{r}_-)$. Now we deal with the equation $f_-(x)=4x^3-3p^2x+q^3=0$. It is clear that the substitution $x=p\operatorname{ch}u$ leads to the equation $p^3\operatorname{ch}3u+q^3=0$ (see $(\mathbf{hr}_-)$). Then $\operatorname{ch}3u=-\frac{q^3}{p^3}$ and it seems that the equalities $(\mathbf{ic}_\pm)$ with $v=-\frac{q^3}{p^3}$ are useful here. Unfortunately, $(\mathbf{ic}_\pm)$ holds only if $|v|\geq 1$. Let us consider first the cases $v\leq -1$ and $v\geq 1$.

4.3.1. The case $(-\frac{q^3}{p^3}\leq -1)$. Keeping in mind that $p>0$, it follows from here that $q>0$ and, hence, the equation $\operatorname{ch}3u=-\frac{q^3}{p^3}$ has no solution, because $\operatorname{ch}3u\geq 1$ for all u. We can avoid this obstacle by putting $x=-p\operatorname{ch}u$. Then one should obtain $-\operatorname{ch}3u=-\frac{q^3}{p^3}$ and, hence, $\operatorname{ch}3u=\frac{q^3}{p^3}\geq 1$. To obtain u one should use $(\mathbf{ic}_+)$:

$$3u=\log_a\left(\frac{q^3}{p^3}+\sqrt{\frac{q^6}{p^6}-1}\right);\ \text{therefore } u=\frac{1}{3}\log_a\frac{q^3+\sqrt{q^6-p^6}}{p^3},$$

so we have

$$u=\log_a\frac{\sqrt[3]{q^3+\sqrt{q^6-p^6}}}{p}=\log_a\lambda,$$

where $\lambda=\frac{\sqrt[3]{q^3+\sqrt{q^6-p^6}}}{p}$. Now we can replace u in $x=-p\operatorname{ch}u$. We have x

$= -\frac{p}{2}(a^u + a^{-u})$ and because, of the identity $a^u = a^{\log_a \lambda} = \lambda$, we obtain the root x: $x = -\frac{p}{2}\left(\lambda + \frac{1}{\lambda}\right)$. Hence

$$x = -\frac{p}{2}\left(\frac{\sqrt[3]{q^3 + \sqrt{q^6 - p^6}}}{p} + \frac{p}{\sqrt[3]{q^3 + \sqrt{q^6 - p^6}}}\right)$$
$$= -\frac{1}{2}\left(\sqrt[3]{q^3 + \sqrt{q^6 - p^6}} + \sqrt[3]{q^3 - \sqrt{q^6 - p^6}}\right).$$

A good application of this formula gives the equation $4x^3 - 6x + 3 = 0$. We have here $p = \sqrt{2}$ and $q = \sqrt[3]{3}$. Now it follows by the above that the only root of this equation is $x_0 = -(\sqrt[3]{4} + \sqrt[3]{2})/2$. A good practice of teaching is to replace x_0 and verify that the expression $4x_0^3 - 6x_0 + 3$ is zero.

Remark. Certainly, we could use (**ic**$_-$). It is easy to see in this case that the root of the equation has the form $x = -\frac{p}{2}(\mu + \frac{1}{\mu})$, where $\mu = \frac{\sqrt[3]{q^3 - \sqrt{q^6 - p^6}}}{p}$. But it follows from the last expression that $\lambda\mu = 1$. Therefore $\mu + \frac{1}{\mu} = \lambda + \frac{1}{\lambda}$ and as is shown above, $x = -\frac{1}{2}(\lambda + \mu)$.

4.3.2. The case $(-\frac{q^3}{p^3} \geq 1)$. In the same manner as in **4.3.1** after the substitution $x = p\,\mathrm{ch}\,u$ the equation (**r**$_-$) takes the form $\mathrm{ch}\,3u = -\frac{q^3}{p^3}$. Solving this equation we get

$$x = \frac{p}{2}\left(\frac{\sqrt[3]{-q^3 + \sqrt{q^6 - p^6}}}{p} + \frac{p}{\sqrt[3]{-q^3 + \sqrt{q^6 - p^6}}}\right)$$
$$= \frac{1}{2}\left(\sqrt[3]{-q^3 + \sqrt{q^6 - p^6}} + \sqrt[3]{-q^3 - \sqrt{q^6 - p^6}}\right).$$

Let us consider for example the equation $4x^3 - 3x - 26 = 0$. Obviously, it has a solution $x = 2$ and below we shall see that it is the only solution. On the other hand, for $p = 1$ and $q = \sqrt[3]{-26}$ the above identity gives

$$x = \frac{1}{2}\left(\sqrt[3]{26 + \sqrt{675}} + \frac{1}{\sqrt[3]{26 + \sqrt{675}}}\right) = \frac{1}{2}\left(\sqrt[3]{26 + \sqrt{675}} + \sqrt[3]{26 - \sqrt{675}}\right)$$

and it is not obvious that the above expression is nothing but 2. Note however, that no harm done if the lecturer use simple calculator to compute this expression: one obtains

$$x_1 = \sqrt[3]{26 + \sqrt{675}} = 3.732\,050\,807\,568\,877\,293\,527\,446\,341\,505\,9\cdots;$$

$$x_2 = \sqrt[3]{26 - \sqrt{675}} = 0.267\,949\,192\,431\,122\,706\,472\,553\,658\,494\,1\ldots$$

and $x_1 + x_2 = 4$ (we use here the standard Windows calculator in scientific mode).

5. The case $\frac{|q|}{|p|} < 1$. The substitution $x = \pm p \operatorname{ch} u$ is not useful here, because the equation $\operatorname{ch} 3u = -\frac{q^3}{p^3}$ has no solution. To realize the problem, let us investigate the function f_-. We have $f'_-(x) = 3(4x^2 - p^2)$. Hence f_- increases in the intervals $\left(-\infty, -\frac{p}{2}\right]$ and $\left[\frac{p}{2}, \infty\right)$. For $x \in \left[-\frac{p}{2}, \frac{p}{2}\right]$ the function f_- is decreasing. The values of f_- in the point of local extremum are $M = f_-\left(-\frac{p}{2}\right) = q^3 + p^3$ and $m = f_-\left(-\frac{p}{2}\right) = q^3 - p^3$. The constant p is positive, that is why $|q| < p$ implies that $M > 0$ and $m < 0$. Therefore, the interval $\left[-\frac{p}{2}, \frac{p}{2}\right]$ contain a root of the equation ($\mathbf{r}_-$). To locate the other roots it is sufficient to see that $f_-(-p) = m$ and $f_-(p) = M$. Thus we obtain that each of the intervals $\left[-p, -\frac{p}{2}\right]$ and $\left[\frac{p}{2}, p\right]$ contains a root. Note in addition, that if $\frac{|q|}{|p|} > 1$ then m and M are both positive or not, hence the equation ($\mathbf{r}_-$) has only one solution.

And so, in case **5.** the equation $f_-(x) = 0$ has three different roots (we leave to the reader the investigation the case when two of them coincides). This is so-called *casus irreducibilis* and it is proved in [1], that it is impossible to find the solutions without using complex numbers.

5.1. Nevertheless, we shall try to say something about solutions. For this purpose, let us make a substitution $x{=}p\cos t$. Then we have $4x^3{-}3p^2x{+}q^3{=}4p^3\cos^3 t{-}3p^3\cos t{+}q^3{=}0$ and therefore $p^3 \cos 3t + q^3 = 0$. And so, we can find an angle $t_0 \in [0, \pi]$, for which $\cos 3t_0 = -\frac{q^3}{p^3}$ since $\frac{|q|^3}{|p|^3} < 1$. Then $x_0 = p \cos t_0$ is a solution. In addition, it is easy to see that the numbers $x_1 = p\cos(t_0 + \frac{2\pi}{3})$ and $x_2 = p\cos(t_0 + \frac{4\pi}{3})$ are also solutions of our equation [2], because $\cos 3t_0 = \cos 3(t_0 + \frac{2k\pi}{3})$ for $k = 1, 2$.

5.2. To write the roots, we should use the inverse function of the cos – this is the function $\arccos x$. It is right that inverse of the trigonometric functions is not a theme for our secondary school, but for students who are familiar one can write that $t_0 = \frac{1}{3} \arccos\left(-\frac{q^3}{p^3}\right)$, hence

$$x_k = \cos\left(\frac{1}{3}\arccos\left(-\frac{q^3}{p^3}\right) + \frac{2k\pi}{3}\right) \text{ for } k = 0, 1, 2.$$

5.3. Examples. We should note, that the consideration from **5.** are completely accessible for the students in high school. A necessary condition for that is to have available calculator. Consider for example the equation $4x^3 - 6x - 1 = 0$. We have here $q = -1$ and $p = \sqrt{2}$. Hence we have to look for an angle t_0, for which $\cos 3t_0 = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}}$. Using the inverse option of the Windows calculator we see that $3t_0 =$ 1.2094292028881888136421330153191... and

$$t_0 = 0.40314306762939627121404433843969\ldots,$$

$$t_1 = t_0 + \frac{2\pi}{3} = 2.4975381700225917635224732606026\ldots \text{ and}$$

$$t_2 = t_0 + \frac{4\pi}{3} = 4.5919332724157872558309021828124\ldots \text{ (radians)}.$$

Next we can write the roots:

$$x_0 = \sqrt{2}\cos t_0 = 1.3008395659415771262321851800939\ldots,$$

$$x_1 = \sqrt{2}\cos t_1 = -1.1309011226299858500660801892384\ldots \text{ and}$$

$$x_2 = \sqrt{2}\cos t_2 = -0.1699384433115912761661049908555 1\ldots.$$

Note, that in our case the intervals $\left[-p, -\frac{p}{2}\right]$, $\left[-\frac{p}{2}, \frac{p}{2}\right]$ and $\left[\frac{p}{2}, p\right]$ coincides with $\left[-\sqrt{2}, -\frac{\sqrt{2}}{2}\right]$, $\left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$ and $\left[\frac{\sqrt{2}}{2}, \sqrt{2}\right]$ since $p = \sqrt{2}$.

5.3.1. Occasionally, it is possible to find the "exact" solutions. Consider as an example the equation $x^3 - 3x + 1 = 0$. To obtain the reduced form, multiply by 4: $4x^3 - 12x + 4 = 0$. Hence, $p = 2$ and $q = \sqrt[3]{4}$. Now we are looking for angle t_0, for which $\cos 3t_0 = -\frac{q^3}{p^3} = -\frac{1}{2}$. Thus, we obtain that $3t_0 = \frac{2\pi}{3}$ and $t_0 = \frac{2\pi}{9}$. Then the solutions of our equation are the numbers $x_0 = 2\cos\frac{2\pi}{9}$, $x_1 = 2\cos\frac{8\pi}{9}$ and $x_2 = 2\cos\frac{14\pi}{9}$.

6. Exotic equations. We finish with an example of cubic equation of the following special type:

$$\textbf{(ex)} \qquad \frac{x^3 + 3p^2x}{p^3 + 3px^2} = \kappa.$$

It is clear that the identities ($\mathbf{t_3}$) suggest to get a substitution $x = p\,\mathrm{th}\,u$ or $x = p\,\mathrm{cth}\,u$. Each such substitution reduces **(ex)** to the form $\mathrm{th}\,3u = \kappa$ or $\mathrm{cth}\,3u = \kappa$. The choice of substitution depends of $|\kappa|$. Clearly for the exclusive values $\kappa = \pm 1$ we have $x = \pm p$. Note, that the exclusive values $\kappa = \pm 1$ we have $x = \pm p$.

We suggest the reader to consider the equation $x^3 - 6x^2 + 3x - 2 = 0$. Here $p = 1$, $\kappa = 2$ and the only solution is $x = \frac{\sqrt[3]{3} + 1}{\sqrt[3]{3} - 1} \approx 5.5223333933593124968516951351377\ldots$. Note, that Cardano's formula gives $x = 2 + \sqrt[3]{3} + \sqrt[3]{9}$.

6.1. Note that the equation **(ex)** has only one root for every $p \neq 0$ and κ, because

$$\left(\frac{x^3 + 3p^2x}{p^3 + 3px^2}\right)' = \frac{3(x^2 - p^2)^2}{(p^2 + 3x^2)^2} > 0 \text{ for } x \neq \pm p.$$

Thus the function $\frac{x^3 + 3p^2x}{p^3 + 3px^2}$ is increasing and the equation **(ex)** has no more than one root.

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РЕШАВАНЕ НА КУБИЧНИ УРАВНЕНИЯ С МЕТОДИ НА АНАЛИЗА

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По традиция темата "кубични уравнения" не се разглежда в нашите средни училища. Вероятно една от причините за това е, че "алгебричната"гледна точка изисква някои познания за комплексните числа. Разбира се, в неразложимия случай това е в известен смисъл неизбежно.
В тази бележка предлагаме "неалгебричен" подход, който позволява преподаването на кубични уравнения в средното училище без да се използва понятието "комплексно число". По наше мнение той е достъпен дори и за скромните възможности, които предлага образователната ни система в наши дни.