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VISCOSITY SOLUTIONS OF NONLINEAR ELLIPTIC AND PARABOLIC EQUATIONS*

Nikolay Kutev

This paper is concerned with viscosity solutions of fully nonlinear, degenerate elliptic and parabolic equations and their applications for some nonclassical pde's problems. The tangential oblique derivative problem for general nonlinear elliptic equations and the global behaviour of the solutions of mean curvature parabolic equations and their connection with some isoparametric problems are considered.

This survey is concerned with viscosity solutions of fully nonlinear, second order degenerate elliptic equations

$$F(x,u,Du,D^2u)=0 \quad \text{in} \quad \Omega \tag{1}$$

including parabolic and first order equations. Here Ω is a bounded domain in $\mathbf{R}^n$, $F(x,r,p,X)$ is a continuous function of all arguments $(x,r,p,X)\in\Omega\times\mathbf{R}\times\mathbf{R}^n\times S^n$ and S^n is the space of all symmetric $n\times n$ matrices. Moreover, the ellipticity condition

$$F(x,r,p,X)\le F(x,r,p,Y) \tag{2}$$

and the monotonicity condition with respect to r

$$\gamma(r-s)\le F(x,r,p,X)-F(x,s,p,X) \tag{3}$$

are satisfied for some $\gamma=$ const >0 and for every $x\in\Omega$; $r,s\in\mathbf{R}$; $p\in\mathbf{R}^n$; $X,Y\in S^n$, whenever $Y\le X$ and $s\le r$.

Viscosity solutions were introduced by M. Crandall and P.-L. Lions [4] for first order Hamilton-Jacobi equations with the method of vanishing viscosity. Later on, the progress in the second order theory was done due to the papers of R. Jensen [12], [13], R. Jensen, P.-L. Lions ans P. Souganidis [14], H. Ishii [10], by means of "supconvolution regularisation", a method which comes from the convex and nonsmooth analysis. The main idea of Jensen is to regularize the semicontinuous sub- and supersolutions of (1), and then to use an appropriate form of the classical maximum principle of Alexandrov [1] or P.-L. Lions [18] in order to prove the uniqueness of the viscosity solutions. The next important step in the existence of the viscosity solution was the adaption of the classical Perron method to semicontinuous sub- and supersolutions of (1) in the paper of H. Ishii [9] (see also [11], [5]).

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The advantages of the method of viscosity solutions are the minimal regularity assumptions on the coefficients of equation (1), boundary data and the domain Ω, as well as the most general ellipticity condition (2). In this way wide class of fully nonlinear degenerate elliptic and parabolic equations, including first order equations, can be investigated. Regarding to the fact that the viscosity solution is only a continuous function, this solution is unique. Moreover, the stability results of the solutions under small perturbations of the coefficients of the equation, boundary data and the domain is a basic achievement of the theory wich is important for the numerical applications.

The original definition of the viscosity solutions comes from the method of vanishing viscosity, i.e. by means of an elliptic regularisation of the equation with a small parameter $\varepsilon > 0$ and a limit of the regularizing parameter ε to zero. This procedure explain the name of the weak solution. Later on, the notion is more close to the comparison principle and the Perron method and there are two equivalent definitions one of them more geometrical and the other more analytical. Roughly speaking, the viscosity solution is a continuous function which is simultaneously a sub- and supersolution of (1).

Definition 1. *An upper semicontinuous function $u \in USC(\Omega)$ is a viscosity subsolution of* (1) *if $F(x_0, u(x_0), D\varphi(x_0), D^2(\varphi(x_0)) \leq 0$ for every $x_0 \in \Omega$ and every C^2 function $\varphi(x)$, such that $u(x_0) = \varphi(x_0)$ and $u(x) \leq \varphi(x)$ in some neighbourhood of x_0.*

An lower semicontinuous function $v \in LSC(\Omega)$ is a viscosity supersolution of (1) if $F(y_0, v(y_0), D\psi(y_0), D^2(\psi(y_0)) \geq 0$ for every $y_0 \in \Omega$ and every C^2 function $\psi(x)$, $\psi(y_0) = v(y_0)$, $\psi(x) \leq v(x)$ in a neighbourhood of y_0.

Finally, a continuous function $w \in C(\Omega)$ is a viscosity solution of (1) if it is both a viscosity sub- and supersolution of (1).

In order to give an equivalent but more analytical definition of the viscosity solutions we define the second order superjets for semicontinuous functions.

Definition 2. *The second order superjet $J^{2,+}_{\Omega}u(x_0)$ is the set of those $(p, X) \in \mathbf{R}^n \times S^n$ for which*

$$u(x) \leq u(x_0) + \langle p, x - x_0\rangle + \frac{1}{2}\langle X(x - x_0), x - x_0\rangle + o(|x - x_0|^2), \quad x \to x_0.$$

Analogously, $J^{2,-}_{\Omega}u(x_0)$ is defined by $J^{2,-}_{\Omega}u(x_0) = J^{2,+}_{\Omega}(-u(x_0))$.

Definition 3. *An upper semicontinuous function $u \in USC(\Omega)$ is a viscosity subsolution of (1) if $F(x_0, u(x_0), p, X) \leq 0$ for every $x_0 \in \Omega$ and for every $(p, X) \in J^{2,+}_{\Omega}u(x_0)$.*

Analogously, the definition of a lower semicontinuous viscosity supersolution $v \in LSC(\Omega)$ is given with the opposite inequality $F(y_0, v(y_0), q, Y) \geq 0$ for every $y_0 \in \Omega$ and for every $(q, Y) \in J^{2,-}_{\Omega}v(y_0)$.

Finally, a continuous function $w \in C(\Omega)$ is a viscosity solution of (1) if it is both a viscosity sub- and supersolution.

We illustrate the advantages of the viscosity solutions with some nonclassical problems for which the standard theory of the pde's is not easy applicable or even does not give any results.

The first example is the tangential oblique derivative problem for the equation (1)

$$B(x, u, Du) = 0 \quad \text{on} \quad \partial\Omega \setminus E \tag{4}$$

where $B(x, u, Du) = \partial u/\partial l + b(x, u)$, $\partial/\partial l = \sum_{i=1}^{n} a^i(x)\partial/\partial x_i$ and $a^i(x) \in C^{1,1}(\overline{\Omega})$,

$b(x,u) \in C^{0,1}(\overline{\Omega} \times \mathbf{R})$. Here the nonzero vector field l is tangential to $\partial\Omega$ at some closed $(n-2)$-dimensional $C^{2,1}$ smooth submanifold E of $\partial\Omega$, but is not tangential to E.

The tangential oblique derivative problem (linear case) was considered for the first time by Poincare in connection with the study of the high and low tides on the surface of the earth. Another application of this problem is in the probability theory (see the references in [23]). Problem (1), (4) is interesting from mathematical point of view even in the linear case because the well known Shapiro-Lopatinskii condition is violated on E and we have a nonclassical elliptic boundary value problem.

It is curious to mention that the viscosity method allows us in a similar way, as for the tangential oblique derivative problem, to investigate the mixed Dirichlet-Neumann boundary conditions (Zaremba problem).

$$B_1(x,u,Du) = 0 \quad \text{on } \partial\Omega, \tag{5}$$

where $B_1(x,u,Du) = \partial u/\partial l + b(x,u)$ on Γ_1, $B_1(x,u,Du) = u - \varphi(x)$ on Γ_2 and $\Gamma_1 = \{x \in \partial\Omega; \langle l(x), \nu(x)\rangle > 0\}$, $\Gamma_1 \cup \Gamma_2 = \partial\Omega$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = E$, $\overline{\Gamma}_2 = \Gamma_2$, where $\nu(x)$ is the unit outher normal to $\partial\Omega$.

In the theory of the tangential oblique derivative problem three cases are possible:

$$\langle l(x), \nu(x)\rangle \ \text{ preserves its sign on } \ \partial\Omega; \tag{6}$$

$$\langle l(x), \nu(x)\rangle \ \text{ changes its sign on } \ \partial\Omega \ \text{ through } \ E \ \text{ from minus to plus in the direction of the vector field } l(x)|_E; \tag{7}$$

$$\langle l(x), \nu(x)\rangle \ \text{ changes its sign on } \partial\Omega \text{ through } \ E \ \text{ from plus to minus in the direction } l(x)|_E. \tag{8}$$

The qualitative properties of the solutions in the above cases are quite different, for example, in the case (7) in order to have uniqueness of the classical solutions for linear elliptic equations some extra condition is necessary on E, i.e.

$$u(x) = \varphi(x) \ \text{ on } \ E \tag{9}$$

while in (8) the solutions are in general discontinuous functions on E (see [23]).

The tangential oblique derivative problem was investigated with energy methods in the Sobolev spaces (see [6], [8], [21], [22], [23]) or with the Schauder technique in the Hölder spaces (see [7], [26], [25], [29], [30]) mainly for linear or weakly nonlinear elliptic and parabolic equations. Unfortunately, the Sobolev technique is applicable only for linear equations, while the Schauder method is usefull for nonlinear problems, but under very restrictive conditions. For example, $C^{3,\alpha}$, $0 < \alpha < 1$, smoothness of the coefficients of the equation, the boundary operator and the domain Ω and uniform ellipticity of the equation is necessary for the a priori estimates in the proof of the existence of a classical solution. Moreover, in both of the cases of Sobolev or Schauder technique, all proofs are based on subelliptic type estimates with precise loss of smoothness of the solutions near E (see [6], [7]) which is possible for uniformly elliptic equations with linear principal part. In contrast to the above methods, the existence of a viscosity solution is based on the Perron method so that the subelliptic estimates of Egorov-Kondratiev are not necessary. In this way, general fully nonlinear, even degenerate elliptic equations can be investigated.

Let us recall the main assumptions guaranteeing a comparison principle for semicontinuous viscosity sub- and supersolutions of (1), (4) with nontangential Neumann's conditions (see [5, th. 7.5] or [11, th. 6.1]).

Suppose that

$$|F(x,r,p,X) - F(x,r,q,Y)| \le \omega(|p-q| + \|X-Y\|) \tag{10}$$

for $x \in \overline{V}$; $p, q \in \mathbf{R}^n$; $X, Y \in S^n$, for some modulus of continuity $\omega(s)$, where V is some oneside neighborhood of $\partial\Omega$;

$$F(y,r,p,Y) - F(x,r,p,X) \le \omega(N|x-y|^2 + |x-y|(|p|+1)) \tag{11}$$

whenever $x, y \in \overline{\Omega}$; $r \in \mathbf{R}$, $p \in \mathbf{R}^n$, $X, Y \in S^n$ and the inequalities

$$-3N \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3N \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \tag{12}$$

hold for every constant $N \ge 1$.

As for the most general first order boundary operator (4) we suppose that $B(x,r,p)$ is monotone with respect to r, i.e.

$$B(x,s,p) \le B(x,r,p) \tag{13}$$

whenever $s \le r$, $x \in \partial\Omega$, $r, s \in \mathbf{R}$, $p \in \mathbf{R}^n$, and B is oriented in the direction of the outer normal $\nu(x)$ to $\partial\Omega$, i.e.

$$B(x,r,p - t\nu(x)) \le B(x,r,p - s\nu(x)) \tag{14}$$

whenever $t \ge s$, $t, s \in \mathbf{R}$, $(x,r,p) \in \partial\Omega \times \mathbf{R} \times \mathbf{R}^n$.

Moreover, for the boundary operator (4) we need a notation of the boundary condition in the viscosity sence which is stable under limit operations with viscosity solutions (see Def. 7.4 in [5]).

Definition 4. *A function $u \in USC(\overline{\Omega})$ is a viscosity subsolution of the boundary condition* (4) *if either $B(x_0, u(x_0), p) \le 0$ or $F(x_0, u(x_0), p, X) \le 0$ for every $x_0 \in \partial\Omega$ and for every $(p, X) \in \overline{J}^{2,+}_{\overline{\Omega}} u(x_0)$.*

A function $v \in LSC(\overline{\Omega})$ is a viscosity supersolution of the boundary condition (4) *if either $B(y_0, v(y_0), q) \ge 0$ or $F(y_0, v(y_0), p, X) \ge 0$ for every $y_0 \in \partial\Omega$ and every $(q, Y) \in \overline{J}^{2,-}_{\overline{\Omega}} v(y_0)$.*

Finally, a continuous function $w \in C(\overline{\Omega})$ is a viscosity solution of the boundary condition (4) *if it is both a sub- and supersolution.*

Now we can formulate the following existence and uniqueness results (see [28, Ths 2.5, 2.6 and 2.8]).

Theorem 1. *Suppose that the condition* (2), (3), (7), (10)−(14) *hold. If $u \in USC(\overline{\Omega})$, $v \in LSC(\overline{\Omega})$ are, resp., bounded sub- and supersolutions of* (1), (4) *and $u \le v$ on E, then $u \le v$ in $\overline{\Omega}$.*

Moreover, if $\underline{u}$, $\overline{u}$, $\underline{u} \le \overline{u}$, are continuous viscosity sub- and supersolutions of (1), (4) *satisfying the same Dirichlet data on E, i.e. $\underline{u} = \overline{u} = \varphi$ on E, then there exists a unique viscosity solution $u \in C(\overline{\Omega})$ such that $\underline{u} \le u \le \overline{u}$ in $\overline{\Omega}$, $u = \varphi$ on E.*

The same result is true for a vector field l satisfying (6). In this case assumptions (11), (12) should be replaces with more precise anisotropic conditions which take into account the direction $l(x)$ on E where the boundary operator is tangential to $\partial\Omega$ (see [28]).

As in Theorem 1, the same result is valid for the Zaremba problem (1), (5) under the same assumptions of Theorem 1.

The second problem which illustrates the adventadges of viscosity solutions and can not be explained with the classical pde's theory is the global behaviour of the solutions to a parabolic mean curvature equation

$$u_t - \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 1 \quad \text{in} \quad Q = \Omega \times (0,\infty), \tag{15}$$

$$u = 0 \quad \text{on} \quad \Gamma = (\partial\Omega \times [0,\infty)) \cup (\Omega \times \{0\}), \tag{16}$$

where Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary.

The global behaviour of the solutions of (15), (16) is deeply connected with the following isoperimetric problem (see p. 196 in [3], [19], [20]).

Given Ω find $\Omega_* \subset \Omega$ which minimizes the ratio of perimeter and the volume, in short

$$h(\Omega) = \min_{G\subset\Omega} \frac{|\partial G|}{|G|} = \frac{|\partial\Omega_*|}{|\Omega_*|} \tag{17}$$

of the smooth simple connected subdomains $G \subset \Omega$, where $h(\Omega)$ and Ω_* are the constant and the domain of Cheger. What is the connection between (15), (16) and (17)?

Marcellini and Miller observed in numerical calculations that a solution of (15), (16) can blow up as time goes to infinity. Those points in Ω in which the speed u_t tends to a maximum (as $t \to \infty$) seem to constitute the set Ω_* which solves (17) and *vice versa.* In particular, solutions of (15), (16) can "detach from the boundary", i.e. fail to satisfy the boundary condition $u = 0$ after some time.

The new phenomena is caused by blow up of the gradient on the boundary, as well as by amplitude blow up of u when Ω is sufficiently large. The combination of these two effects leads to a traveling wave phenomenon or detachment of solutions on part of the boundary and the development of "a rising elliptic cap" depending on the mean curvature of the minimizing set Ω_*. We illustrate the situation only for the square, $\Omega = K = \{0 < x_i < a, i = 1, 2\}$ (see [15, Th. 2.3]).

Theorem 2. *Let K be the square with length $a > 2 + \sqrt{\pi}$. Then*

i) Problem (15), (16) has a unique solution $u \in C^\infty(K \times (0,\infty)) \cap C(\overline{K} \times [0,\infty))$ which solves the Dirichlet condition (16) in the viscosity sense. Moreover, the trace of u on the boundary is Lipschitz continuous.

ii) The gradient of the solution blows up on $\partial K_1 \cap \partial K$, after a finite time $T_*(x)$. Here K_1 is obtained from K by rounding off the corners with circular arcs of radius 1. Until this blow up occurs, (16) holds in the classical sense. After the blow up time $T_*(x)$ the solution detaches from the boundary data on $\partial K_1 \cap \partial K$ with infinite slope.

iii) If K_R denotes "the square with rounded corners of radius R", then for $R \in [1, a/(2+\sqrt{\pi})]$ the following sharp estimates hold for large time

$$u(x,t) \geq \left(1 - \frac{1}{R}\right) t + \underline{\omega}_R(x) \quad \textit{for} \;\; x \in K_R, \;\; t \gg 0.$$

$$u(x,t) \leq \left(1 - \frac{1}{R}\right) t + \overline{\omega}_R(x) \quad \textit{for} \;\; x \in K \setminus K_R, \;\; t \gg 0,$$

where $\underline{\omega}_R$ and $\overline{\omega}_R$ are independent of t and locally finite.

Theorem 2 confirms the conjectures of Marcellini and Miller in several ways. On the set Ω_*, defined in (17), the solution u grows with maximasl speed, and off the set Ω_* it grows less than maximal in time.

Let us finish this survey with some open problems in the theory of the viscosity solutions. The main question which comes from the applications is how to define a discontinuous viscosity solution which is unique and stable under small perturbations of the equation, boundary data and the domain? For example, case (8) in the tangential oblique derivative problem leads to jump discontinuous solutions, even in the linear case. The other example is the image process equation

$$u_t - \text{ div } (a(|\nabla u|^2)\nabla u) = 0 \qquad \text{in } \Omega \times (0,T), \tag{18}$$

$$a(|\nabla u|^2)\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \times (0,T), \quad u(x,0) = u_0(x) \text{ on } \Omega,$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain with boundary of class C^1 and ν is the exterior normal to $\partial\Omega$. The structural assumptions on $a \in C^1([0,\infty))$ are

$$a(s) > 0, \qquad b(s) := a(s) + 2sa'(s) \text{ is positive for } s \text{ near } 0 \tag{19}$$

$$\text{and changes sign exactly once at } s_0^2 > 0.$$

Typical examples of such diffusion fuctions a are $a(s) = e^{-s}$ or $a(s) = (1+s)^{-1}$. They are used in image enhancement processes, see [24]. The function $u_0(x)$ represents the brightness of a picture which one wants to denoise. Numerical computations have shown that equation (18) can produce the desired effect that $u(x,T)$ provides a sharper image than $u(x,0)$.

For the time being, the existence of a weak solution of problem (18) is an open problem. It seems that the solutions are jump discontinuous functions. The Perron method is not applicable for (18) because there is no comparison principle for semicontinuous viscosity sub- and supersolutions. In fact Theorem 4.1 in [16] guarantees a conditional comparison principle between C^1 smooth weak solutions of (18) which is only enough for the uniqueness of C^1 weak solutions. Without comparison principle, in general, Perron method produces discontinuous solutions. Our conjecture is that at the points x where the Perron solution $u(x)$ is discontinuous, the set valued map $x \to [u_*(x), u^*(x)]$ should be considered. Here the upper and lower semicontinuous envelops $u^*(x)$ and $u_*(x)$ are defined as

$$u^* = \lim_{r\to 0} \sup\{u(y); |y-x| < r, \ y \in \Omega\},$$

$$u_* = \lim_{r\to 0} \inf\{u(y); |y-x| < r, \ y \in \Omega\}.$$

This notion allows for an extension of the comparison principle and stability properties for discontinuous solutions which is a major feature from the theory of viscosity solutions.

The second open problem is the nature of the important conditions (11), (12) for the comparison principle. They naturally appear in the method of the proof, but there are many questions about their necessity or how to find some other criteria guaranteing the validity of the comparison principle. Our main conjecture is that they are deeply connected with the conditions for interior gradient estimates of continuous viscosity solutions.

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Nikolay Kutev
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia, Bulgaria

ВИСКОЗНИ РЕШЕНИЯ ЗА НЕЛИНЕЙНИ ЕЛИПТИЧНИ И ПАРАБОЛИЧНИ УРАВНЕНИЯ

Николай Кутев

Работата е посветена на теория на вискозните решения за напълно нелинейни, израждащи се елиптични и параболични уравнения и тяхното приложение за някои некласически задачи. Разгледана е задачата с тангенциална наклонена производна за най-общи нелинейни елиптични уравнения и глобалното поведение на решенията на нестационарното уравнение на повърхнини с предварително зададена средна кривина, както и връзката с някои изопараметрични задачи.