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CONVEXITY OF THE OBJECTIVE FUNCTIONAL ON SOME OPTIMAL CONTROL PROBLEMS*

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The paper suggests a sufficient condition for convexity of the objective functional of some optimal control problems. It is written on the basis of Mangasarian sufficient condition for optimality. The condition for convexity of the functional can be used to prove the convergence of a minimizing sequence to an optimal solution when optimal control problem is solved numerically.

1. Introduction. The convex functionals have some important for the optimal control theory properties. It is known, that if a convex functional has a point of local minimum on a convex set, then that local minimum is a global one. In the numerical methods of the extremal problems the convexity of the objective functional is a condition for establishment of convergence and for estimation of the velocity of the convergence of a minimizing sequence. Let is considered a problem for finding of a minimum of a convex functional on a bounded closed convex subset of a reflexive Banach space. For this problem it is known, that every minimizing sequence is weakly convergent to the set of the solutions [3, p. 53]. Besides, if the minimizing sequence is obtained on the basis of some gradient method, then there is an estimation of the velocity of the convergence of the objective functional to its minimal value [3, p. 70, p. 75, p. 80].

Our aim in this paper is to find criteria for convexity of the objective functional in the optimal control problems. Our basic result is the sufficient condition, which is given as Theorem 3 of Section 4. Some additional results are formulated without proofs in the conclusion.

In the paper we denote by $x(\cdot)$ the relation $t \to x(t)$, where t is the argument, and $x(t)$ is the value of the function. The scalar product of the vectors x and y we write as (x, y), but in the space R^n – simply as $x\,y$. The symbol $\square$ we use to indicate the end of a proof.

2. Statement of the main problem. We shall consider the following optimal control problem: Minimize the functional

$$J(u(\cdot)) = \int_0^T f_0(x, t, u)dt \qquad (1)$$

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subjected to

$$\dot{x} = f(x, t, u), \qquad x(0) = x_0 \tag{2}$$

$$u(\cdot) \in U \subset L_2[0, T], \qquad U \text{ is a convex set} \tag{3}$$

Here the state vector $x \in R^n$, the control vector $u \in R^r$, and the phase speed $f(x, t, u) \in R^n$. The terminal time T and the state x_0 are fixed, and the terminal state is free.

We want to emphasize, that the problem (1)–(3) consists of finding a conditional extremum of the functional J on a subset U of a Hilbert space. The value of the functional depends only on the element $u(\cdot) \in U$ and is calculated by the formulas (1) and (2).

For solving the problem (1)–(3) the following Hamilton's function is used[1]:

$$H(x, t, u, p) = f_0(x, t, u) + pf(x, t, u) \tag{4}$$

We say, that the trajectory $x(\cdot)$ corresponds to the control $u(\cdot)$, if it is solution of the Cauchy problem (2), in which u is the given control. The adjoint vector $p(\cdot)$ corresponds to $x(\cdot)$ and $u(\cdot)$, if it is solution of the next adjoint Cauchy problem:

$$\dot{p} = -\frac{\partial H(x, t, u, p)}{\partial x}, \qquad p(T) = 0 \tag{5}$$

in which x and u are the given trajectory and control, respectively.

3. Some auxiliary results. From now on we use some conditions for convexity and for reaching an extremum of a functional. The next two theorems are well known:

Theorem 1. *Let $U \subset H$ be a nonempty convex subset of the Hilbert space H, and J be a continuously differentiable in Frechet sense functional.*

A necessary and sufficient condition for convexity of J in U is for each two elements $u, u + \delta u \in U$ to be fulfilled the inequality

$$J(u + \delta u) \geq J(u) + (J'(u), \delta u), \tag{6}$$

where $J'(u)$ is the Frechet derivative of the functional $J(u)$.

Theorem 2. *Let the condition of the Theorem 1 be fulfilled. Besides, let J be a convex functional. Then, the functional J reaches its minimum at the point $u \in U$ if and only if*

$$(J'(u), v - u) \geq 0 \quad \forall v \in U. \tag{7}$$

These theorems can be found in [2], on page 103 and page 109, respectively. It is necessary the notion function to be replaced by functional only. **Theorem 1** is formulated in [3, p. 24] also. In the monograph [2] this theorem is proved only for open convex sets under weaker conditions, while in [3] it is given for arbitrary convex sets.

4. The main result. In the next theorem we suggest a sufficient condition for convexity of the objective functional of the problen (1)–(3), obtained on the basis of the Mangasarian sufficient condition for optimality [1, p. 105].

Theorem 3. *Assume that f and f_0 are jointly continuous of their arguments together with their partial derivatives with respect to x and u. Besides, let f, $\partial f/\partial x$, $\partial f/\partial u$, $\partial f_0/\partial x$ and $\partial f_0/\partial u$ be Lipschitzian functions of x and u.*

[1] In some bibliographic sources as [3], [4] etc, this function is introduced and used with the opposite sign.

For the convexity of the objective functional J it is sufficient the Hamilton's function H to be jointly convex in (x, u) for each solution p of the adjoint Cauchy's problem (5), which corresponds to some control $v \in U$.

Proof. Under the hypothesis of the theorem, the objective functional is differentiable. The Frechet derivative is [3, p. 93]:

$$J'(u(\cdot)) = \frac{\partial H(x(\cdot), t, u(\cdot), p(\cdot))}{\partial u}, \tag{8}$$

where x is the corresponding to u trajectory, and p is the corresponding to x and u adjoint variable. Besides, the derivative is continuous with respect to the variable u. In order to prove the convexity of J, it is sufficient to prove that for each two controls $u, u + \delta u \in U$ the inequality (6) is fulfilled. In other words, we must prove the inequality

$$J(u + \delta u) \geq J(u) + \int_0^T \frac{\partial H(x, t, u, p)}{\partial u} \delta u \, dt \tag{9}$$

The proof of this inequality looks like the suggested in [1, p. 104] proof of Mangasarian theorem. Let H is jointly convex by (x, u) for every $t \in [0, T]$ and for every adjoint variable p, which corresponds to some control $v \in U$. According to Theorem 1 the inequality

$$H(x + \delta x, t, u + \delta u, p) - H(x, t, u, p) \geq \frac{\partial H(x, t, u, p)}{\partial x} \delta x + \frac{\partial H(x, t, u, p)}{\partial u} \delta u$$

is fulfilled.

This inequality is fulfilled when x corresponds to u, p corresponds to (x, u) and $x + \delta x$ corresponds to $u + \delta u$. By integrating this inequality and using the equation (5) we obtain

$$\begin{aligned} \int_0^T (H(x + \delta x, t, u + \delta u, p) - H(x, t, u, p)) \, dt &\geq -\int_0^T \dot{p} \delta x \, dt + \int_0^T \frac{\partial H}{\partial u} \delta u \, dt \\ &= -p\delta x \Big|_0^T + \int_0^T p \delta \dot{x} dt + \int_0^T \frac{\partial H}{\partial u} \delta u \, dt \end{aligned}$$

The first term in the right-hand side is equal to zero, because $p(T) = 0$ and $\delta x(0) = 0$. By using that $\delta \dot{x} = f(x + \delta x, t, u + \delta u) - f(x, t, u)$, we obtain

$$\begin{aligned} &\int_0^T (H(x + \delta x, t, u + \delta u, p) - H(x, t, u, p)) \, dt \geq \\ &\geq \int_0^T p(f(x + \delta x, t, u + \delta u) - f(x, t, u)) \, dt + \int_0^T \frac{\partial H}{\partial u} \delta u \, dt \end{aligned} \tag{10}$$

Using the definition of the Hamiltonian we can give an expression of the function f_0 as a function of independent variables. This expression is

$$H(x, t, u, p) - pf(x, t, u) = f_0(x, t, u).$$

By using this expression, from the inequality (10) we obtain

$$\int_0^T f_0(x+\delta x, t, u+\delta u)\, dt - \int_0^T f_0(x, t, u)\, dt \geq \int_0^T \frac{\partial H}{\partial u} \delta u\, dt \tag{11}$$

which actually is the inequality (9). Therefore, the sufficient condition is proved. □

If we use the above suggested theorem, we are able to prove immediately the Mangasarian sufficient condition, that was already mentioned.

Corollary (The Mangasarian theorem). *Assume that for the control u^* the conditions from the Pontryagin's minimum principle*[2] *are fulfilled. In other words, u^* is an admissible control for the problem (1) − (3) and the Hamilton's function $H(x,t,u,p)$ reaches its minimum at u^* for the respective solutions of (2) for the trajectory, and of (5) for the adjoint variable. Besides, assume that there are fulfilled the conditions of Theorem 3 and Hamilton's function $H(x,t,u,p)$ is jointly convex by (x,u) for every $t \in [0,T]$ and every adjoint variable p that corresponds to some control $v \in U$. Then, u^* is an optimal control.*

Proof. Since H contains his minimum, according to Theorem 2 there is fulfilled the inequality

$$\int_0^T \frac{\partial H(x^*, t, \psi^*, u^*)}{\partial u} \delta u\, dt \geq 0$$

From it and from the inequality (11) it follows the truth of the corollary. □

5. Conclusion. The sufficient condition for convexity, which we suggested in **Theorem 3**, is used for more wide class of optimal control problems. Let be given a problem of Bolza with fixed time and with convex sets of the initial and the terminal states. A sufficient condition for convexity of the objective functional of this problem is the Hamiltonian to be jointly convex by (x,u), the terminal term to be convex and the adjoint variable to satisfy a transversality condition. As a particular case of our sufficient conditions we may consider the condition suggested in the example in [3, p. 34]. But there is a requirement the phase trajectories to be solutions of a linear by (x,u) system of differential equations.

Analogous sufficient conditions can be suggested for the strong convexity of the objective functional of the optimal control problems.

As we noticed, the considered sufficient conditions are obtained on the basis of the Mangasarian sufficient condition for optimality. Its application is limited by the requirement for convexity of the Hamiltonian. Unfortunately, this requirement can not be relaxed by deriving sufficient conditions on the basis of the Arrow's condition [1, p. 107].

These additional results are available to the author. We don't include their proofs for the sake of the limited size of the paper.

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[2] It is known as the maximum principle, see the note 1.

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ИЗПЪКНАЛОСТ НА ЦЕЛЕВИЯ ФУНКЦИОНАЛ В НЯКОИ ЗАДАЧИ НА ОПТИМАЛНОТО УПРАВЛЕНИЕ

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В настоящата публикация е предложено достатъчно условие за изпъкналост на целевия функционал за някои задачи на оптималното управление. То е получено на основата на достатъчното условие на Мангасарян за оптималност. Условието за изпъкналост на функционала може да се използва за установяване на сходимостта на минимизираща редица към оптимално решение при числено решаване на задача на оптималното управление