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# A CHARACTERIZATION OF DEVELOPABLE TWO-DIMENSIONAL SURFACES IN EUCLIDEAN SPACE*

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We prove that each developable two-dimensional surface in Euclidean space is a surface with flat normal connection. We give a characterization of the developable two-dimensional surfaces in terms of surfaces with flat normal connection.
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A ruled two-dimensional surface $M^2$ in $n$-dimensional Euclidean space $\mathbb{E}^n$ is a one-parameter system $\{\mathbb{E}^1(v)\}, v \in J$ of one-dimensional linear subspaces $\mathbb{E}^1(v)$ of $\mathbb{E}^n$, defined in an interval $J \subset \mathbb{R}$. The lines $\mathbb{E}^1(v)$ are called generators of $M^2$. A ruled surface $M^2 = \{\mathbb{E}^1(v)\}, v \in J$ is said to be *developable*, if the tangent space $T_pM^2$ at all regular points $p$ of an arbitrary fixed generator $\mathbb{E}^1(v)$ is one and the same.

Each ruled surface $M^2 = \{\mathbb{E}^1(v)\}, v \in J$ in $\mathbb{E}^n$ can be parametrized as follows:

$$z(u,v) = x(v) + u\,e(v), \quad u \in \mathbb{R},\ v \in J, \tag{1}$$

where $x(v)$ and $e(v)$ are vector-valued functions, defined in $J$, such that the vectors $e(v)$ and $x'(v) + u\,e'(v)$ are linearly independent for all $v \in J$. The tangent space of $M^2$ is spanned by the vectors

$$z_u = e(v);$$
$$z_v = x'(v) + u\,e'(v).$$

Using that for a developable surface $M^2$ the space $\operatorname{span}\{z_u, z_v\}$ is constant at the points of a fixed line $\mathbb{E}^1(v)$, we obtain that the ruled surface, defined by (1), is developable if and only if the vectors $e(v)$, $e'(v)$ and $x'(v)$ are linearly dependent.

**Proposition 1.** *Each developable two-dimensional surface in Euclidean space $\mathbb{E}^n$ is a surface with flat normal connection.*

**Proof.** Let $M^2$ be a developable surface, defined by equality (1), where $e(v)$, $e'(v)$ and $x'(v)$ are linearly dependent vectors. Without loss of generality we assume that $e^2(v) = 1$. Then, the vector fields $e(v)$ and $e'(v)$ are orthogonal and the tangent space of $M^2$ is $\operatorname{span}\{e(v), e'(v)\}$. Since $x'(v) \in \mathit{span}\{e(v), e'(v)\}$, then $x'(v)$ is presented by

$$x'(v) = p(v)\,e(v) + q(v)\,e'(v)$$

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for some functions $p(v)$ and $q(v)$. Hence, the tangent space of $M^2$ is spanned by

$$\begin{aligned} z_u &= e; \\ z_v &= p\, e + (u+q)\, e'. \end{aligned}$$

Considering only the regular points of $M^2$ (where $u \neq -q$), we choose an orthonormal tangent frame field $\{X_1, X_2\}$ in the following way:

$$\begin{aligned} X_1 &= e = z_u; \\ X_2 &= \frac{e'}{\sqrt{(e')^2}} = -\frac{p}{(u+q)\sqrt{(e')^2}}\, z_u + \frac{1}{(u+q)\sqrt{(e')^2}}\, z_v. \end{aligned} \tag{2}$$

Since the tangent space of $M^2$ does not depend on the parameter $u$, then the normal space of $M^2$ is spanned by vector fields $b_1(v), \dots, b_{n-2}(v)$. With respect to the basis $\{e(v), e'(v), b_1(v), \dots, b_{n-2}(v)\}$ of $\mathbb{E}^n$ the derivatives of $b_\alpha,\ \alpha = 1, \dots, n-2$ are decomposed in the form

$$b'_\alpha = c_\alpha\, e' + c_\alpha^\beta\, b_\beta, \tag{3}$$

where $c_\alpha$ and $c_\alpha^\beta,\ \alpha, \beta = 1, \dots, n-2$ are functions of $v$. Here and further on the summation convention is assumed.

Let $\nabla'$ be the Levi-Civita connection of the standard metric $\langle , \rangle$ in $\mathbb{E}^n$. We denote by $D$ the normal connection of $M^2$. Using (2) and (3) we get

$$\begin{aligned} \nabla'_{X_1} b_\alpha &= 0; \\ \nabla'_{X_2} b_\alpha &= \frac{c_\alpha}{u+q}\, X_2 + \frac{c_\alpha^\beta}{(u+q)\sqrt{(e')^2}}\, b_\beta; \end{aligned} \qquad \alpha = 1, \dots, n-2. \tag{4}$$

Having in mind the Weingarten formula, we obtain

$$\begin{aligned} D_{X_1} b_\alpha &= 0; \\ D_{X_2} b_\alpha &= \frac{c_\alpha^\beta}{(u+q)\sqrt{(e')^2}}\, b_\beta; \end{aligned} \qquad \alpha = 1, \dots, n-2. \tag{5}$$

The normal curvatures $R_{b_\alpha},\ \alpha = 1, \dots, n-2$ of $M^2$, corresponding to the normal vector fields $b_\alpha,\ \alpha = 1, \dots, n-2$, are given by

$$R_{b_\alpha}(X_1, X_2) = D_{X_1} D_{X_2} b_\alpha - D_{X_2} D_{X_1} b_\alpha - D_{[X_1, X_2]} b_\alpha, \quad \alpha = 1, \dots, n-2.$$

Using (2) we calculate

$$[X_1, X_2] = -\frac{1}{u+q}\, X_2. \tag{6}$$

From (5) and (6) we obtain

$$R_{b_\alpha}(X_1, X_2) = 0, \quad \alpha = 1, \dots, n-2,$$

which implies that $M^2$ is a surface with flat normal connection.

**Remark:** Each two-dimensional plane $\mathbb{E}^2$ in $\mathbb{E}^n$ can be considered as a trivial developable surface. Obviously, each plane $\mathbb{E}^2$ is a surface with flat normal connection.

Our aim is to characterize the two-dimensional surfaces with flat normal connection, which are developable surfaces.

Let $M^2$ be a two-dimensional surface in $\mathbb{E}^n$ with flat normal connection. According to [1] locally there exist $n-2$ mutually orthogonal unit normal vector fields $b_1, \dots, b_{n-2}$, which are parallel in the normal bundle. Moreover, there exist two mutually orthogonal unit tangent vector fields $X_1$ and $X_2$ on $M^2$, such that with respect to the frame

$X_1, X_2, b_1, \dots, b_{n-2}$ the shape operators $A_{b_\alpha}$, corresponding to $b_\alpha$, $\alpha = 1, \dots, n-2$, are given by [2]

$$A_{b_\alpha} = \begin{pmatrix} \kappa_1^\alpha & 0 \\ 0 & \kappa_2^\alpha \end{pmatrix}, \quad \alpha = 1, \dots, n-2,$$

where $\kappa_1^\alpha$ and $\kappa_2^\alpha$, $\alpha = 1, \dots, n-2$ are functions on $M^2$. The vector fields $X_1$ and $X_2$ determine the principal directions of $M^2$. The derivative formulas of $M^2$ with respect to the frame $X_1, X_2, b_1, \dots, b_{n-2}$ look like

$$(7) \qquad \begin{aligned} \nabla'_{X_1} X_1 &= \phantom{-f_2 X_1} f_1\, X_2 + \kappa_1^\alpha\, b_\alpha; \\ \nabla'_{X_1} X_2 &= -f_1\, X_1; \\ \nabla'_{X_2} X_1 &= \phantom{-f_2 X_1} -f_2\, X_2; \\ \nabla'_{X_2} X_2 &= f_2\, X_1 \phantom{-f_2 X_2} + \kappa_2^\alpha\, b_\alpha, \end{aligned}$$

where $f_1 = \langle \nabla'_{X_1} X_1, X_2 \rangle$; $f_2 = \langle \nabla'_{X_2} X_2, X_1 \rangle$ and

$$(8) \qquad \begin{aligned} \nabla'_{X_1} b_\alpha &= -\kappa_1^\alpha\, X_1; \\ \nabla'_{X_2} b_\alpha &= -\kappa_2^\alpha X_2; \end{aligned} \qquad \alpha = 1, \dots, n-2.$$

If $\bar{b}_1, \dots, \bar{b}_{n-2}$ is another normal frame field, consisting of parallel normal vector fields of $M^2$, then $b_1, \dots, b_{n-2}$ and $\bar{b}_1, \dots, \bar{b}_{n-2}$ are connected by a constant orthogonal matrix.

Obviously, the following lemma holds true.

**Lemma 2.** *Let $M^2$ be a two-dimensional surface in $\mathbb{E}^n$ with flat normal connection. $M^2$ is locally a plane if and only if $\kappa_i^\alpha = 0,\ i = 1, 2,\ \alpha = 1, \dots, n-2$.*

We shall describe the two-dimensional surfaces in $\mathbb{E}^n$ with flat normal connection, for which $\kappa_1^\alpha \kappa_2^\alpha = 0,\ \alpha = 1, \dots, n-2$ for each parallel normal frame field.

**Lemma 3.** *Let $M^2$ be a two-dimensional surface in $\mathbb{E}^n$ with flat normal connection and $\kappa_1^\alpha \kappa_2^\alpha = 0,\ \alpha = 1, \dots, n-2$ for each parallel normal frame field. Then, there exists a neighborhood $U \subset M^2$, such that $\kappa^\alpha_{1\,|U} = 0$ for all $\alpha = 1, \dots, n-2$ (or $\kappa^\alpha_{2\,|U} = 0$ for all $\alpha = 1, \dots, n-2$).*

**Proof.** Let $M^2$ be a surface with flat normal connection and $X_1, X_2, b_1, \dots, b_{n-2}$ be a frame field of $M^2$, satisfying (8), where $\kappa_1^\alpha \kappa_2^\alpha = 0,\ \alpha = 1, \dots, n-2$.

If there exist a point $p \in M^2$ and an index $\alpha \in \{1, \dots, n-2\}$, such that $\kappa_1^\alpha(p) \neq 0$, then there exists a neighborhood $U$ of $p$, in which $\kappa_1^\alpha \neq 0$. Hence, $\kappa^\alpha_{2\,|U} = 0$. We shall prove that $\kappa^\beta_{2\,|U} = 0$ for all $\beta = 1, \dots, n-2$. Suppose that there exist a point $q \in U$ and an index $\beta \neq \alpha$, such that $\kappa_2^\beta(q) \neq 0$. Then, there exists a neighborhood $U_1 \subset U,\ q \in U_1$, such that $\kappa^\beta_{2\,|U_1} \neq 0$. Hence, $\kappa^\beta_{1\,|U_1} = 0$. Without loss of generality we assume that $\alpha = 1$, $\beta = 2$ (up to numeration). Let us consider the parallel normal frame field $\{\bar{b}_1, \bar{b}_2, b_3, \dots, b_{n-2}\}$, where

$$\bar{b}_1 = \frac{b_1 + b_2}{\sqrt{2}}; \quad \bar{b}_2 = \frac{b_1 - b_2}{\sqrt{2}}.$$

Then, for the parallel normal vector fields $\bar{b}_1$ and $\bar{b}_2$ we have

$$\bar{\kappa}_1^1 = \frac{\kappa_1^1 + \kappa_1^2}{\sqrt{2}}; \quad \bar{\kappa}_2^1 = \frac{\kappa_2^1 + \kappa_2^2}{\sqrt{2}}; \quad \bar{\kappa}_1^2 = \frac{\kappa_1^1 - \kappa_1^2}{\sqrt{2}}; \quad \bar{\kappa}_2^2 = \frac{\kappa_2^1 - \kappa_2^2}{\sqrt{2}}.$$

Hence, $\bar{\kappa}_1^\alpha \bar{\kappa}^\alpha_{2\,|U_1} \neq 0,\ \alpha = 1, 2$, which contradicts the condition $\kappa_1^\alpha \kappa_2^\alpha = 0,\ \alpha = 1, \dots, n-2$ for each parallel normal frame field.

Consequently, $\kappa^{\beta}_{2\,|U} = 0$ for each $\beta = 1, \dots, n-2$.

We give a characterization of the developable two-dimensional surfaces in the following theorem.

**Theorem 4.** *Let $M^2$ be a two-dimensional surface in $\mathbb{E}^n$ with flat normal connection. Then, $M^2$ is locally a developable surface if and only if $\kappa_1^{\alpha}\kappa_2^{\alpha} = 0$, $\alpha = 1, \dots, n-2$ for each parallel normal frame field.*

**Proof.** Let $M^2$ be a surface with flat normal connection and $X_1, X_2, b_1, \dots, b_{n-2}$ be a frame field of $M^2$, for which formulas (7) and (8) hold good.

I. Suppose that $\kappa_1^{\alpha}\kappa_2^{\alpha} = 0$, $\alpha = 1, \dots, n-2$. We shall prove that locally $M^2$ is either a plane or a non-trivial developable surface.

According to Lemma 3, $\kappa_2^{\alpha} = 0$ for all $\alpha = 1, \dots, n-2$ in a neighborhood $U \subset M^2$. Let $p$ be an arbitrary point of $U$ and $c_2$ be the integral curve of $X_2$, passing through $p$. Since $\nabla'_{X_2} b_\alpha = 0,\ \alpha = 1, \dots, n-2$, then the normal space $\operatorname{span}\{b_1, \dots, b_{n-2}\}$ of $M^2$ is constant at the points of $c_2$ and hence, the tangent space $\operatorname{span}\{X_1, X_2\}$ of $M^2$ at the points of $c_2$ is one and the same. Using the derivative formulas (7) of $M^2$ we get

$$\begin{aligned} \nabla'_{X_2} X_2 &= \ f_2\, X_1; \\ \nabla'_{X_2} X_1 &= -f_2\, X_2. \end{aligned} \tag{9}$$

Let the curve $c_2$ be parametrized by $x = x(v),\ v \in J$ and $x'(v) = t = X_2$. Then, by the Frenet's formulas of $c_2$ and (9) we get that the curvature of $c_2$ is $\kappa = \pm f_2$ and the principal normal is $n = \pm X_1$.

If $f_2 = 0$, i.e. $\kappa = 0$, then $c_2$ is a straight line. Consequently, for each point $p \in U$ there exists a straight line passing through $p$, i.e. $M^2$ is locally a ruled surface. Moreover, since the tangent space of $M^2$ at the points of each line is one and the same, then $M^2$ is locally a developable surface.

If $f_2(q) \neq 0$ at a point $q \in U$, then there exists a neighborhood $\widetilde{U} \subset U,\ q \in \widetilde{U}$, such that $f_{2\,|\widetilde{U}} \neq 0$. From the equalities (9) we get

$$\begin{aligned} t' &= \kappa\, n; \\ n' &= -\kappa\, t, \end{aligned}$$

i.e. $c_2$ is a plane curve, lying in its osculating plane $\operatorname{span}\{t, n\} = \operatorname{span}\{X_1, X_2\}$. Consequently, for each point $p \in U$ there exists a plane curve, passing through $p$, i.e. $M^2$ is locally a one-parameter system $\{c_2(u)\}, u \in I$ of plane curves $c_2(u)$, defined in an interval $I \subset \mathbb{R}$. Let for a fixed $u \in I$ the curve $c_2(u)$ lie in a plane $\mathbb{E}^2(u)$ spanned by the vectors $e_1(u)$ and $e_2(u)$. Hence, the surface $M^2$ has a local parametrization, given by

$$z(u, v) = a_1(v)\, e_1(u) + a_2(v)\, e_2(u), \quad u \in I,\ v \in J \tag{10}$$

for some functions $a_1(v)$ and $a_2(v)$. Then, the tangent space of $M^2$ is spanned by

$$\begin{aligned} z_u &= a_1(v)\, e_1'(u) + a_2(v)\, e_2'(u); \\ z_v &= a_1'(v)\, e_1(u) + a_2'(v)\, e_2(u). \end{aligned}$$

Using that the tangent space of $M^2$ is $\operatorname{span}\{e_1(u), e_2(u)\}$, we obtain that $e_1'(u), e_2'(u) \in \operatorname{span}\{e_1(u), e_2(u)\}$, which implies that $\mathbb{E}^2(u)$ is a constant plane $\mathbb{E}^2_0$. Let $e_1^0 = const,\ e_2^0 = const$ be an orthonormal basis of $\mathbb{E}^2_0$. Then,

$$\begin{aligned} e_1(u) &= \cos\varphi(u)\, e_1^0 + \sin\varphi(u)\, e_2^0; \\ e_2(u) &= -\sin\varphi(u)\, e_1^0 + \cos\varphi(u)\, e_2^0 \end{aligned} \tag{11}$$

for some function $\varphi(u)$. Using (10) and (11) and setting $z^1(u,v) = a_1(v)\cos\varphi(u) - a_2(v)\sin\varphi(u)$; $z^2(u,v) = a_1(v)\sin\varphi(u) + a_2(v)\cos\varphi(u)$, we obtain the following local parametrization of $M^2$:

$$z(u,v) = z^1(u,v)\, e_1^0(u) + z^2(u,v)\, e_2^0(u). \tag{12}$$

The equality (12) implies that $M^2$ lies in the plane $\mathbb{E}_0^2$, i.e. $M^2$ is locally a plane.

II. Let $M^2$ be a developable surface, defined by (1), where $e(v)$, $e'(v)$ and $x'(v)$ are linearly dependent vectors. We consider the frame field $\{X_1, X_2, b_1, \dots, b_{n-2}\}$ of $M^2$, determined in the same way as in the proof of Proposition 1. So, formulas (4) hold good. Let $N_1, \dots, N_{n-2}$ be a normal frame field of $M^2$, consisting of parallel normal vector fields. Using (4) we obtain

$$\begin{aligned} \nabla'_{X_1} N_\alpha &= 0; \\ \nabla'_{X_2} N_\alpha &= \frac{\sigma_\alpha^\beta\, c_\beta}{u+q}\, X_2; \end{aligned} \qquad \alpha = 1, \dots, n-2,$$

where $\sigma_\alpha^\beta = \langle N_\alpha, b_\beta \rangle$; $\alpha, \beta = 1, \dots, n-2$. Hence,

$$\varkappa_1^\alpha = 0; \quad \varkappa_2^\alpha = -\frac{\sigma_\alpha^\beta\, c_\beta}{u+q}; \quad \alpha = 1, \dots, n-2.$$

Consequently, $\varkappa_1^\alpha \varkappa_2^\alpha = 0$, $\alpha = 1, \dots, n-2$.

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## ЕДНА ХАРАКТЕРИЗАЦИЯ НА РАЗВИВАЕМИТЕ ДВУМЕЛНИ ПОВЪРХНИНИ В ЕВКЛИДОВО ПРОСТРАНСТВО


**Величка В. Милушева**



Доказваме, че всяка развиваема двумелна повърхнина в $n$-мерно евклидово пространство има плоска нормална свързаност. Даваме характеризацията на развиваемите двумерни повърхнини на езика на повърхнините с плоска нормална свързаност.