МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2006
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2006
Proceedings of the Thirty Fifth Spring Conference of
the Union of Bulgarian Mathematicians
Borovets, April 5–8, 2006

IMPROVEMENT OF GRAPH THEORY WEI'S INEQUALITY*

Nedyalko D. Nenov

Wei in [8] and [9] discovered a bound on the clique number of a given graph in terms of its degree sequence. In this note we give an improvement of this result.

We consider only finite non-oriented graphs without loops and multiple edges. A set of p vertices of a graph is called a p-clique if each two of them are adjacent. The greatest positive integer p for which G has a p-clique is called clique number of G and is denoted by $\mathrm{cl}(G)$. A set of vertices of a graph is independent if the vertices are pairwise nonadjacent. The independence number $\alpha(G)$ of a graph G is the cardinality of a largest independent set of G.

In this note we shall use the following notations:

- $V(G)$ is the vertex set of graph G;
- $N(v)$, $v \in V(G)$ is the set of all vertices of G adjacent to v;
- $N(V)$, $V \subseteq V(G)$ is the set $\bigcap_{v\in V} N(v)$;
- $d(v)$, $v \in V(G)$ is the degree of the vertex v, i.e. $d(v) = |N(v)|$.

Let G be a graph, $|V(G)| = n$ and $V \subseteq V(G)$. We define

$$W(V) = \sum_{v\in V} \frac{1}{n-d(v)};$$
$$W(G) = W(V(G)).$$

Wei in [8] and [9] discovered the inequality

$$\alpha(G) \geq \sum_{v\in V(G)} \frac{1}{1+d(v)}.$$

Applying this inequality to the complementary graph of G we see that it is equivalent to the following inequality

$$\mathrm{cl}(G) \geq \sum_{v\in V(G)} \frac{1}{n-d(v)},$$

*Key words: clique number, degree sequence
2000 Mathematics Subject Classification: 05C35

that is

$$\mathrm{cl}(G) \geq W(G). \tag{1}$$

Alon and Spencer [1] gave an elegant probabilistic proof of Wei's inequality. In the present note we shall improve the inequality (1).

Definition 1. *Let G be a graph, $|V(G)| = n$ and $V \subseteq V(G)$. The set V is called a δ-set in G, if*

$$d(v) \leq n - |V|$$

for all $v \in V$.

Example 1. Any independent set V of vertices of a graph G is a δ-set in G since $N(v) \subseteq V(G) \setminus V$ for all $v \in V$.

Example 2. Let $V \subseteq V(G)$ and $|V| \geq \max\{d(v), v \in V(G)\}$. Since $d(v) \leq |V|$ for all $v \in V(G)$, $V(G) \setminus V$, is a δ-set in G.

The next statement obviously follows from Definition 1.

Proposition 1. *Let V be a δ-set in a graph G. Then $W(V) \leq 1$.*

Definition 2. *A graph G is called an r-partite graph if*

$$V(G) = V_1 \cup \cdots \cup V_r, \quad V_i \cap V_j = \emptyset, \quad i \neq j,$$

where the sets V_i, $i = 1, \ldots, r$, are independent. If the sets V_i, $i = 1, \ldots, r$, are δ-sets in G, then G is called generalized r-partite graph. The smallest integer r such that G is a generalized r-partite graph is denoted by $\varphi(G)$.

Proposition 2. $\varphi(G) \geq W(G)$.

Proof. Let $\varphi(G) = r$ and

$$V(G) = V_1 \cup \cdots \cup V_r, \quad V_i \cap V_j = \emptyset, \quad i \neq j,$$

where V_i, $i = 1, \ldots, r$, are δ-sets in G. Since $V_i \cap V_j = \emptyset$, $i \neq j$, we have

$$W(G) = \sum_{i=1}^{r} W(V_i).$$

According to Proposition 1, $W(V_i) \leq 1$, $i = 1, \ldots, r$. Thus $W(G) \leq r = \varphi(G)$.

Below (see Theorem 1) we shall prove that $\mathrm{cl}(G) \geq \varphi(G)$. Thus (1) follows from Proposition 2.

Definition 3 [2]**.** *Let G be a graph and $v_1, \ldots, v_r \in V(G)$. The sequence $v_1, \ldots, v_r$ is called an α-sequence in G if the following conditions are satisfied:*

(i) $d(v_1) = \max\{d(v) \mid v \in V(G)\}$*;*

(ii) $v_i \in N(v_1, \ldots, v_{i-1})$ *and* v_i *has maximal degree in the graph* $G[N(v_1, \ldots, v_{i-1})]$, $2 \leq i \leq r$.

Every α-sequence $v_1, \dots, v_s$ in the graph G can be extended to an α-sequence $v_1, \dots, v_s, \dots, v_r$ such that $N(v_1, \dots, v_{r-1})$ be a δ-set in G. Indeed, if the α-sequence $v_1, \dots, v_s, \dots, v_r$ is such that it is not continued in a $(r+1)$-clique (i.e. $v_1, \dots, v_s, \dots, v_r$ is a maximal α-sequence in the sense of inclusion) then $N(v_1, \dots, v_{r-1})$ is an independent set and, therefore, a δ-set in G. However, there are α-sequences $v_1, \dots, v_r$ such that $N(v_1, \dots, v_{r-1})$ is a δ-set but it is not an independent set.

Theorem 1. *Let G be a graph and $v_1, \dots, v_r$, $r \geq 2$, be an α-sequence in G such that $N(v_1, \dots, v_{r-1})$ is a δ-set in G. Then*
(a) $\varphi(G) \leq r \leq \mathrm{cl}(G)$;
(b) $r \geq W(G)$.

Proof. According to Definition 3, $v_1, \dots, v_r$ is an r-clique and thus $r \leq \mathrm{cl}(G)$. Since $N(v_1, \dots, v_{r-1})$ is a δ-set, the graph G is a generalized r-partite graph, [6]. Hence $r \geq \varphi(G)$. The inequality (b) follows from (a) and Proposition 2.

Remark. Theorem 1 (b) was proved in [7] in the special case when $N(v_1, \dots, v_{r-1})$ is independent set in G.

Definition 4. *Let G be a graph and $v_1, \dots, v_r \in V(G)$. The sequence $v_1, \dots, v_r$ is called β-sequence in G if the following conditions are satisfied:*

(i) $d(v_1) = \max\{d(v) \mid v \in V(G)\}$;

(ii) $v_i \in N(v_1, \dots, v_{i-1})$ *and* $d(v_i) = \max\{d(v) \mid v \in N(v_1, \dots, v_{r-1})\}$, $2 \leq i \leq r$.

Theorem 2. *Let $v_1, \dots, v_r$ be a β-sequence in a graph G such that*

$$d(v_1) + \cdots + d(v_r) \leq (r-1)n,$$

where $n = |V(G)|$. Then $r \geq W(G)$.

Proof. According to [5], it follows from $d(v_1) + \cdots + d(v_r) \leq (r-1)n$, that G is a generalized r-partite graph. Hence $r \geq \varphi(G)$ and Theorem 2 follows from Proposition 2.

Corollary. *Let G be a graph, $|V(G)| = n$ and $v_1, \dots, v_r$ be a β-sequence in G which is not contained in $(r+1)$-clique. Then $r \geq W(G)$.*

Proof. Since $v_1, \dots, v_r$ is not contained in $(r+1)$-clique it follows that $d(v_1) + \cdots + d(v_r) \leq (r-1)n$, [3].

Theorem 3. *Let G be a graph, $|V(G)| = n$ and $v_1, \dots, v_r$, $r \geq 2$, be a β-sequence in G such that $N(v_1, \dots, v_{r-1})$ is a δ-set in G. Then $r \geq W(G)$.*

Proof. Since $N(v_1, \dots, v_{r-1})$ is a δ-set according to [6] there exists an r-partition

$$V(G) = V_1 \cup \cdots \cup V_r, \qquad V_i \cap V_j = \emptyset, \quad i \neq j,$$

where V_i, $i = 1, \dots, r$, are δ-sets and $v_i \in V_i$. Thus, we have

$$d(v_i) \leq n - |V_i|, \quad i = 1, \dots, r.$$

Summing up these inequalities we obtain that $d(v_1) + \cdots + d(v_r) \leq (r-1)n$. Therefore Theorem 3 follows from Theorem 2.

REFERENCES

[1] N. Alon, J. H. Spencer. The Probabilistic Method. Wiley, New York, 1992.
[2] N. Khadzhiivanov, N. Nenov. Extremal problems for s-graphs and a theorem of Turan. *Serdica Math. J.*, **3** (1977), 117–125 (in Russian).
[3] N. Khadzhiivanov, N. Nenov. Sequences of maximal degree vertices in graphs. *Serdica Math. J.*, **30** (2004), 95–102.
[4] N. Khadzhiivanov, N. Nenov. Generalized r-partite graphs and Turan's Theorem. *Compt. rend. Acad. bulg. Sci.*, **57**, *2* (2004), 19–24.
[5] N. Khadzhiivanov, N. Nenov. Saturated β-sequences in graphs. *Compt. rend. Acad. bulg. Sci.*, **57**, *6* (2004), 49–54.
[6] N. Khadzhiivanov, N. Nenov. Balanced vertex sets in graph. *Ann. Univ. Sofia, Fac. Math. Inf.*, **97** (2005), 50–64.
[7] O. Murphy. Lower bounds on the stability number of graph computed in terms of degree. *Discrete Math.*, **90** (1991), 207–211.
[8] V. K. Wei. A lower bound on the stability number of a simple graph. Bell Laboratories Technical Memorandum, 81–11217–9. Murray Hill, NJ, 1981.
[9] V. K. Wei. Coding for a Multiple Access Channel. Ph. D. Thesis, University of Hawaii, Honolulu, 1980.

Nedyalko Dimov Nenov
Faculty of Mathematics and Informatics
St Kliment Ohridski University of Sofia
5, James Bourchier Blvd
1164 Sofia, Bulgaria
e-mail: nenov@fmi.uni-sofia.bg

ПОДОБРЕНИЕ НА НЕРАВЕНСТВОТО НА WEI ОТ ТЕОРИЯ НА ГРАФИТЕ

Недялко Д. Ненов

Wei в [8] и [9] получава оценка за кликовото число на граф чрез степените на върховете му. В тази работа ние подобряваме този резултат.