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OSCILLATIONS OF A CLASS OF EQUATIONS AND INEQUALITIES OF FOURTH ORDER*

Zornitza A. Petrova

We establish sufficient conditions for oscillation of a wide class of ordinary differential equations of fourth order. So that, we begin with sufficient conditions for the absence of eventually positive solutions of proper ordinary differential inequalities. We extend a rezult of Kusano and Yoshida concerned with such an inequality.

There is extensive literature devoted to the oscillation behaviour of linear and nonlinear ordinary differential equations and inequalities with or without delays. The excellent monographies of Györi and Ladas [1], Ladde, Lakshmikantham and Zhang [3], Shevelo [5] and Mishev and Bainov [4] consist large lists of articles.

We obtain sufficient conditions for oscillation of the equation

$$z^{iv}(t) + mz''(t) + g(z(t), z'(t), z''(t), z'''(t)) = f(t), \tag{1}$$

where $m = \text{const} \in \mathbf{R}$, $f(t) \in C([T, \infty), \mathbf{R})$, $T \geq 0$ is a large enough constant and $g(z, \xi, \eta, \zeta) \in C(\mathbf{R}^4, \mathbf{R})$ satisfies the conditions:

$$\begin{aligned} g(z,\xi,\eta,\zeta) \geq qz, \quad \forall\ (z,\xi,\eta,\zeta) \in \mathbf{R}_+ \times \mathbf{R}^3, \\ g(z,\xi,\eta,\zeta) \leq qz, \quad \forall\ (z,\xi,\eta,\zeta) \in \mathbf{R}_- \times \mathbf{R}^3, \end{aligned} \tag{2}$$

where $q = const > 0$. We prove the present theorems via our conclusions for the absence of eventually positive solutions of the inequality:

$$z^{iv}(t) + mz''(t) + g(z(t), z'(t), z''(t), z'''(t)) \leq f(t) \tag{3}$$

under the same assumptions. We mention that we could rewrite our results for (3) as respective results for the absence of eventually negative solutions of

$$z^{iv}(t) + mz''(t) + g(z(t), z'(t), z''(t), z'''(t)) \geq f(t). \tag{4}$$

Then we apply them to find sufficient conditions for oscillation of the equation (1).

We point out that (2) guarantees that every positive solution of (3) in a set $M \subseteq \mathbf{R}$ is also a positive solution of

$$z^{iv}(t) + mz''(t) + qz(t) \leq f(t) \tag{5}$$

in the same set. Simillarly, every negative solution of (4) in $M \subseteq \mathbf{R}$ is also a negative solution of

$$z^{iv}(t) + mz''(t) + qz(t) \geq f(t) \tag{6}$$

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in M.

The considerations for the equation:

$$z^{iv}(t) + mz''(t) + qz(t) = f(t) \tag{7}$$

make us sure that all the results are sharp.

We shall finish this paper with:

$$z^{iv}(t) + mz''(t) + qz(t) + \lambda z(t-\tau) \leq f(t), \tag{8}$$

$$z^{iv}(t) + mz''(t) + qz(t) + \lambda z(t-\tau) \geq f(t) \tag{9}$$

and

$$z^{iv}(t) + mz''(t) + qz(t) + \lambda z(t-\tau) = f(t), \tag{10}$$

where we suppose additionaly that $\lambda \in \mathbf{R}$ and $\tau \geq 0$ are constants.

All the sufficient conditions for oscillation here are based on the following lemma of Yoshida [6]. We suppose that $L = const > 0$ and $\rho = \text{const} \geq 0$ everywhere.

Lemma 1. ([6]) *If there is a number $s \geq \rho$ such that*

$$\int_s^{s+\pi/L} F(t) \sin L(t-s) dt \leq 0, \tag{11}$$

then the ordinary differential inequality

$$z''(t) + L^2 z(t) \leq F(t) \tag{12}$$

has no positive solution in $(s, s+\pi/L]$.

Definition 1. *We say that the function $\varphi(t) \in C([c,\infty), \mathbf{R})$ is* **oscillating** *when $t \to \infty$, if there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that*

$$\lim_{n\to\infty} t_n = \infty \qquad \textit{and} \qquad \varphi(t_n) = 0.$$

Definition 2. *We say that the function $\varphi(t) \in C([c,\infty), \mathbf{R})$ is* **eventually positive** (**eventually negative**), *if there exists $\tilde{c} = const \geq c$ such that*

$$\varphi(t) > 0 \quad (\varphi(t) < 0), \qquad \forall\ t \in [\tilde{c}, \infty).$$

The next corollary was not formulated in [6] but it was applied there.

Let define

$$\Phi(s) = \int_s^{s+\pi/L} F(t) \sin L(t-s) dt. \tag{13}$$

Corollary 1. *If the function $\Phi(s)$ is eventually nonpositive, then the ordinary differential inequality (12) has no eventually positive solution.*

Corollary 2. *The ordinary differential inequality*

$$z''(t) + L^2 z(t) \leq 0 \tag{14}$$

has no eventually positive solution.

The result of Corollary 2 was extracted in [8] as a particular case of the paper [2]. It gurantees that all the above results are sharp.

Let us explain the connection between the inequalities of fourth order at the begining and these one of second order here. Yoshida and Kusano considered Timoshenko beam equation:

$$\frac{\partial^4 u(x,t)}{\partial t^4}+\alpha\beta\gamma\frac{\partial^2 u(x,t)}{\partial t^2}-(\beta+\gamma)\frac{\partial^4 u(x,t)}{\partial x^2\partial t^2}+\beta\gamma\frac{\partial^4 u(x,t)}{\partial x^4}+c(x,t,u)=f(x,t) \tag{15}$$

in [8], where α, β and γ are positive constants. They followed the classical method in the oscillation theory of partial differential equations there. In particular, it means that they established a pair of fourth order ordinary differential inequalities corresponding to the average of positive solution as well as of the negative one in [8]. After a proper integration by parts they found such a relation for these averages, which helped them to apply Corollary 2. Obviously, the equation (15) is a non-homogeneous one and the inequality (14) is a homogeneous one. This fact makes the proofs complicated. Later, Yoshida [6] investigated hyperbolic equations via Lemma 1, which was proved there. Since the inequality (12) is a non-homogeneous one then the same author applied it for (15) in the mentioned way in [7]. There were not any common results for fourth order ordinary differential equations nor in [7] neither in [8].

We apply

$$\Phi_4(\theta)=\int_\theta^{\theta+\pi/\tilde{L}}\int_s^{s+\pi/L} f(t)\sin L(t-s)\sin\tilde{L}(s-\theta)dtds,\quad \tilde{L}=\sqrt{m-L^2} \tag{16}$$

instead of $\Phi(s)$ in all the bellow. We point out that Yoshida [7] obtained an unique function of the above type. In fact, we find a family of functions $\Phi_4(\theta)$, which depend on the constant L. This is very important since these functions give information for the distributions of the zeros of (1) and a whole family allow us to apply computers. Moreover, there was a restriction on the numbers m and q in [7], which is omitted here.

Theorem 1. *Assume that* (2) *holds and that the positive constants* L *and* q *satisfy the conditions:*

$$L\in(0,\sqrt{m})\quad\text{and}\quad q+(L^2-m)L^2>0. \tag{17}$$

If there is a number $s\geq\rho$ *such that*

$$\Phi_4(s+\pi/L)\leq 0, \tag{18}$$

then the ordinary differential inequalities (3) *and* (5) *have no positive solution in* $(s,s+\pi/L+\pi/\tilde{L}]$. *Further, if the function* $\Phi_4(\theta)$ *is eventually nonpositive, then the ordinary differential inequalities* (3) *and* (5) *have no eventually positive solution.*

Proof. It is enough to consider (5) only. We suppose to the contrary, i. e.

$$z(t)>0,\quad \forall\, t\in(s,s+\pi/L+\pi/\tilde{L}]. \tag{19}$$

Then we obtain a contradiction with Lemma 1. More exactly, from one hand we have that the function

(20) $\zeta(s)=z(s+\pi/L)+z(s)$ is positive solution of the inequality

$$z''(s)+\tilde{L}^2z(s)\leq\tilde{F}(s),\quad\text{in}\quad(s+\pi/L,s+\pi/L+\pi/\tilde{L}], \tag{21}$$

where

$$\tilde{F}(s)=\frac{1}{L}\int_s^{s+\pi/L} f(t)\sin L(t-s)dt \tag{22}$$

but from the other hand, all the conditions of Lemma 1 are fulfilled for (21) since (18) is the present particular case of (11).

Obviously, we obtain that

$$\zeta(s) > 0, \quad \forall\, t \in (s, s + \pi/L + \pi/\tilde{L}] \tag{23}$$

directly from (19) and (20). Now we shall show that $\zeta(s)$ is solution of (21). Since $z(t)$ satisfies (5) in $(s, s + \pi/L + \pi/\tilde{L}]$ then

$$\int_s^{s+\pi/L} \big(z^{iv}(t) + mz''(t) + qz(t)\big) \sin L(t-s)dt \le \int_s^{s+\pi/L} f(t) \sin L(t-s)dt \tag{24}$$

and we replace the following integrations by parts:

$$\int_s^{s+\pi/L} z''(t) \sin L(t-s)dt = L(z(s+\pi/L) + z(s)) - L^2 \int_s^{s+\pi/L} z(t) \sin L(t-s)dt,$$

$$\int_s^{s+\pi/L} z^{iv}(t) \sin L(t-s)dt = L(z''(s+\pi/L) + z''(s)) - L^2 \int_s^{s+\pi/L} z''(t) \sin L(t-s)dt$$

there. Also, (17) guarantees that

$$\big(q + (L^2 - m)L^2\big) \int_s^{\tilde{s}} z(t) sin L(t-s)dt \ge 0. \tag{25}$$

Then the contrary assumption leads to the fact that $\zeta(s)$ is such that:

$$L\left(\zeta''(s) + (m - L^2)\zeta(s)\right) \le \int_s^{s+\pi/L} f(t) sin L(t-s)dt. \tag{26}$$

We apply Corollary 1 to finish the second part of the theorem.

Theorem 2. *Let* (2) *and* (17) *be satisfied. If the function* $\Phi_4(s)$ *is oscillating, then every solution of the ordinary differential equations* (1) *and* (7) *oscillates.*

Proof. Everything is based on Theorem 1 as well as on the inequalities (5) and (6). More exactly, every solution of the equation (7) is also solution of (5) and (6) since if $z^{iv}(t) + mz''(t) + qz(t) = f(t)$, then obviously

$$z^{iv}(t) + mz''(t) + qz(t) \le f(t) \quad \text{and} \quad z^{iv}(t) + mz''(t) + qz(t) \ge f(t).$$

Moreover, every solution of the equation (1) is also solution of (5) and (6) since if (1) holds then also

$$z^{iv}(t) + mz''(t) + qz(t) \le z^{iv}(t) + mz''(t) + g(z(t), z'(t), z''(t), z'''(t)) = f(t),$$

if $z(t) > 0$,

$$z^{iv}(t) + mz''(t) + qz(t) \ge z^{iv}(t) + mz''(t) + g(z(t), z'(t), z''(t), z'''(t)) = f(t),$$

if $z(t) < 0$.

Here we prove much more. In fact, we establish a sufficient condition for the distribution of the zeros of (1) and (7) in the case, where the function $\Phi_4(s)$ is oscillating. It means that there exists a sequence $\{s_n\}_{n=1}^{\infty}$:

$$\lim_{n\to\infty} s_n = \infty \ \text{ and } \ \Phi(s_n + \pi/L) = 0, \ \text{ i. e. } \ \Phi(s_n + \pi/L) \le 0 \ \text{ and } \ \Phi(s_n + \pi/L) \ge 0.$$

We apply Theorem 1 to conclude that these equations have nor positive solution in $(s_n, s_n + \pi/L + \pi/\tilde{L}]$ neither negative solution in $(s_n, s_n + \pi/L + \pi/\tilde{L}]$, $\forall\, n \in N$. It means just that (1) and (7) has at least one zero in $(s_n, s_n + \pi/L + \pi/\tilde{L}]$, $\forall\ n \in N$ and we finish the proof.

Theorem 3. *Let* (2) *and* (17) *be fufilled and let there is a number* $s \geq \rho$ *such that* (18) *holds. If* $\lambda \geq 0$ *then* (8) *has no positive solution in* $(s-\tau, s+\pi/L+\pi/\tilde{L}]$. *If* $\lambda \in [-\left(q+(L^2-m)L^2\right), 0]$ *then* (8) *has no positive and monotonically increasing solution in* $(s-\tau, s+\pi/L+\pi/\tilde{L}]$.

Further, let the function $\Phi_4(\theta)$ *be eventually nonpositive. If* $\lambda \geq 0$ *then* (8) *has no eventually positive solution. If* $\lambda \in [-\left(q+(L^2-m)L^2\right), 0]$ *then the ordinary differential inequality* (8) *has no eventually positive and monotonically increasing solution.*

Proof. It is enough to investigate the case, where $t \in (s-\tau, s+\pi/L+\pi/\tilde{L}]$. We suppose to the contrary again. Our aim is to explain that (26) is fufilled in both situations because it allows us to apply Lemma 1. Since here we have

$$\int_s^{s+\pi/L} \left[z^{iv}(t) + mz''(t) + qz(t) + \lambda z(t-\tau)\right] \sin L(t-s)dt \leq \int_s^{s+\pi/L} f(t) sinL(t-s)dt$$

instead of (24) then the unique difficult moment is to see that if $\lambda \in [-\left(q+(L^2-m)L^2\right), 0]$ then the contrary assumption gives us that

$$z(t) \geq z(t-\tau) \quad \text{and} \quad q+(L^2-m)L^2 \geq -\lambda \geq 0.$$

Hence,

$$\left(q+(L^2-m)L^2\right) z(t) \geq -\lambda z(t) \geq -\lambda z(t-\tau).$$

Theorem 4. *Let* (2) *and* (17) *be fulfilled and let the function* $\Phi_4(s)$ *be oscillating. If* $\lambda \geq 0$ *then every solution of the equation (10) oscillates. If* $\lambda \in [-\left(q+(L^2-m)L^2\right), 0]$ *then the equation* (10) *has nor eventually positive and monotonically increasing solution neither eventually negative and monotonically decreasing solution.*

Proof. Everything follows from Theorem 3.

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Zornitza A. Petrova
Faculty of Applied Mathematics and Informatics
Technical University of Sofia
Sofia, Bulgaria
e-mail: zap@tu-sofia.bg

ОСЦИЛАЦИИ НА КЛАС ОТ УРАВНЕНИЯ И НЕРАВЕНСТВА ОТ ЧЕТВЪРТИ РЕД

Зорница А. Петрова

Установяваме достатъчни условия за осцилация на широк клас от обикновени диференциални уравнения от четвърти ред. Започваме с достатъчни условия за отсъствието на финално положителни решения на подходящи обикновени диференциални неравенства. Разширяваме резултат на Йошида и Кусано, свързан с това неравенство.