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ON BALLOONS, MEMBRANES AND SURFACES REPRESENTING THEM*

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The well known Laplace-Young equation asserts that the pressure difference across the film or a membrane in a equilibruim is proportional to the mean curvature with a proportionality constant the surface tension of the interface. Here we present two variants of this equation leading to the surfaces of Delaunay and the mylar balloon and in this way provide their nonvariational characterization.

1. Introduction. The equilibrium conditions for axisymmetric membrane lead to highly nonlinear equations that sometimes can be solved exactly to give the shape. The quantities that play the most crucial role in this sort of problems are membrane weight density, circumferential and meridional stresses and the differential pressure. Besides in Biology membrane models have found many nice applications in other scientific and technological areas such as the design and production of large scientific balloons used by space agencies to carry out research in the upper stratosphere. Guided by mechanical ideas we derive two classes of shapes having quite interesting geometrical properties.

2. Axisymmetric membranes. In order to parametrize an axisymmetric membrane we need to define its generating profile curve. Let $s \longrightarrow (u(s), v(s))$ be such a curve in some meridional slice where s is the natural parameter provided by the corresponding arclength. The total arclength is denoted by L. We parametrize the membrane surface $\mathcal{S}$ in the ordinary Euclidean space $\mathbb{R}^3$ with a fixed orthonormal basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ by making use of s and the angle φ specifying the rotation of the XOY plane *via* the vector-valued function

$$\mathbf{x}(s,\varphi) = u(s)\mathbf{e}_1(\varphi) + v(s)\mathbf{e}_3(\varphi), \qquad 0 < s \le L, \quad 0 \le \varphi < 2\pi. \tag{1}$$

Here the vector $\mathbf{e}_1(\varphi)$ is the new position of $\mathbf{i}$ after the rotation given by

$$\mathbf{e}_1(\varphi) = \cos\varphi\mathbf{i} + \sin\varphi\mathbf{j}\,. \tag{2}$$

The second vector in the equation (1) is along the axis around which the rotation takes place and, therefore, remains constant, i.e., $\mathbf{e}_3(\varphi) = \mathrm{const} = \mathbf{k}$. The set $\{\mathbf{e}_1, \mathbf{e}_3\}$ can be completed to an orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of $\mathbb{R}^3$ by a third vector $\mathbf{e}_2(\varphi)$ which is introduced as a cross product of the vectors $\mathbf{e}_3(\varphi)$ and $\mathbf{e}_1(\varphi)$, i.e.,

$$\mathbf{e}_2(\varphi) = \mathbf{e}_3(\varphi) \times \mathbf{e}_1(\varphi) = \mathbf{k} \times \mathbf{e}_1(\varphi) = -\sin\varphi\mathbf{i} + \cos\varphi\mathbf{j}\,.$$

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In order to determine some important characteristics specifying the membrane's shape, we calculate the derivatives of $\mathbf{x}(s,\varphi)$. The derivative with respect to s gives us the tangent vector at each point along the generating curve

$$\mathbf{t}(s,\varphi)=\mathbf{x}_s(s,\varphi)=u'(s)\mathbf{e}_1(\varphi)+v'(s)\mathbf{k}.$$

In view of what follows we introduce also $\theta(s)$, which measures the angle spanned between the tangent vector $\mathbf{t}$ and $\mathbf{k}$. The coordinates $u(s)$ and $v(s)$ depend on $\theta(s)$ by the equations

$$u'(s)=\sin\theta(s) \tag{3}$$

$$v'(s)=\cos\theta(s). \tag{4}$$

Using (3) and (4), the equation for the tangent vector becomes

$$\mathbf{t}(s,\varphi)=\sin\theta(s)\mathbf{e}_1(\varphi)+\cos\theta(s)\mathbf{k}. \tag{5}$$

The second derivative with respect to the parameter s of $\mathbf{x}(s,\varphi)$ is

$$\mathbf{x}_{ss}=\theta'(s)\cos\theta(s)\mathbf{e}_1(\varphi)-\theta'(s)\sin\theta(s)\mathbf{k}. \tag{6}$$

Next, we find the first and second order derivatives of $\mathbf{x}(s,\varphi)$ with respect to φ

$$\mathbf{x}_\varphi = u(s)(\mathbf{e}_1(\varphi))_\varphi=u(s)\mathbf{e}_2(\varphi) \tag{7}$$
$$\mathbf{x}_{\varphi\varphi} = u(s)(\mathbf{e}_2(\varphi))_\varphi=-u(s)\mathbf{e}_1(\varphi) \tag{8}$$

and, finally, the mixed derivative

$$\mathbf{x}_{s\varphi}=\mathbf{x}_{\varphi s}=\sin\theta(s)\mathbf{e}_2(\varphi). \tag{9}$$

We need also to find out the outward normal $\mathbf{n}(s,\varphi)$. It can be presented as a cross product of the tangent vector $\mathbf{t}(s,\varphi)$ and $\mathbf{e}_2(\varphi)$, i.e.,

$$\mathbf{n}(s,\varphi)=\mathbf{t}(s,\varphi)\times\mathbf{e}_2(\varphi)=-\cos\theta(s)\mathbf{e}_1(\varphi)+\sin\theta(s)\mathbf{k}. \tag{10}$$

By means of the coefficients of the first fundamental form

$$E=\mathbf{x}_s^2=1,\qquad F=\mathbf{x}_s.\mathbf{x}_\varphi=0,\qquad G=\mathbf{x}_\varphi^2=u^2(s), \tag{11}$$

and those of the second fundamental form of $\mathcal{S}$,

$$L=\mathbf{n}.\mathbf{x}_{ss}=-\theta'(s),\quad M=\mathbf{n}.\mathbf{x}_{s\varphi}=0,\quad N=\mathbf{n}.\mathbf{x}_{\varphi\varphi}=u(s)\cos\theta(s). \tag{12}$$

one can find easily the mean curvature H (meaning average) of the membrane surface under consideration.

Making use of the standard formula for H in the textbooks on classical differential geometry (see e.g. [10])

$$H=\frac{LG-2MF+NE}{2(EG-F^2)}$$

we end up with the following result

$$H = -\frac{1}{2}\left(\theta'(s) - \frac{\cos\theta(s)}{u(s)}\right). \tag{13}$$

3. Equilibrium equations. Let us consider now the forces acting on the membrane surface. The internal forces are

$$\mathbf{f}_1(s,\varphi) = \sigma_m(s)\mathbf{t}(s,\varphi) \quad \text{and} \quad \mathbf{f}_2(s,\varphi) = \sigma_c(s)\mathbf{e}_2(\varphi). \tag{14}$$

In the left-hand side of equation (14) σ_m means the meridional stress resultant and in the right-hand one σ_c is the circumferential stress resultant (for more details see [1]). Let us mention that the situation when $\sigma_c \equiv 0$ is referred in ballooning literature as the natural shape model.
The external forces depend on the pressure and the density of the membrane's material, namely,

$$\mathbf{f}(s,\varphi) = -p(s)\mathbf{n}(s,\varphi) - w(s)\mathbf{k}. \tag{15}$$

Here $p(s)$ is the hydrostatic differential pressure and $w(s)$ is the weight density of the film. Balancing the internal and external forces we are led to the following equilibrium equations

$$(\sigma_m u(s)\mathbf{t})_s - \sigma_c\mathbf{e}_1(\varphi) + u(s)\mathbf{f}(s,\varphi) = 0. \tag{16}$$

We can project the above vectorial equation onto $\mathbf{n}$ and $\mathbf{t}$ and this gives us respectively

$$(\sigma_m u(s))\frac{\mathrm{d}\theta}{\mathrm{d}s} = \sigma_c\cos\theta(s) - w(s)u(s)\sin\theta(s) - p(s)u(s) \tag{17}$$

$$\frac{\mathrm{d}(\sigma_m u(s))}{\mathrm{d}s} = \sigma_c\sin\theta(s) + w(s)u(s)\cos\theta(s). \tag{18}$$

4. Shapes and related surfaces. In the period between 1960 and 1970 J. Smalley did an extensive work on axisymmetric balloon shapes and implement these models on a digital computer (all relevant references on the subject can be found in [1]).
As most of Smalley's considerations were of numerical origin it deserve to look for those models possessing analytical solutions. Despite that the system governing these shapes is highly nonlinear we have been successful in finding a few exact solutions which are presented below.
Let us start with the case when we can neglect the film weight contribution, i.e., we suppose that $w(s) \equiv 0$, and, hence, in such a case we have instead the equations (17) and (18) the system

$$(\sigma_m u(s))\frac{\mathrm{d}\theta}{\mathrm{d}s} = \sigma_c\cos\theta(s) - p(s)u(s) \tag{19}$$

$$\frac{\mathrm{d}(\sigma_m u(s))}{\mathrm{d}s} = \sigma_c\sin\theta(s). \tag{20}$$

In order to be coherent with the geometrical relation (3), the last equation implies that the meridional and circumferential stresses are constant and of the same magnitude, i.e., $\sigma_m = \sigma_c = \sigma =$ constant, while (19) specifies the mean curvature of $\mathcal{S}$, namely

$$H = \frac{p(s)}{2\sigma}. \tag{21}$$

As it can be recognized immediately, if we can arrange that the hydrostatic pressure is also a constant, i.e., $p(s) = p_o =$ constant, then we end up with a surface of constant mean curvature

$$H = \frac{p_o}{2\sigma} = \text{constant}. \tag{22}$$

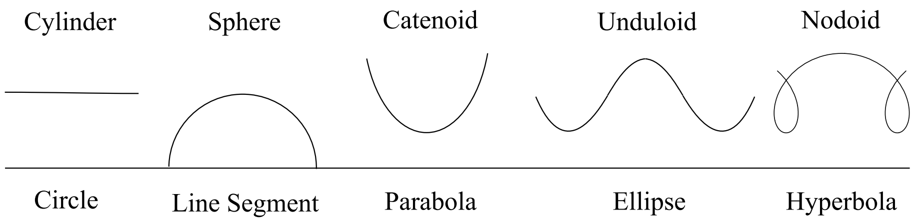

Fig. 1. The profile curves of Delaunay's surfaces obtained by rolling the conics listed below the horizontal line on it.

Delaunay [3] has isolated this class of surfaces guided by a genuine geometrical argument – all they are just the traces of the foci of the non-degenerate conics when they roll along a straight line in a plane (*roulettes* in French). In an Appendix to the same paper Sturm characterizes Delaunay's surfaces variationally as those surfaces of revolution having a minimal lateral area at a fixed volume. That in turn revealed why these surfaces make their appearance as soap bubbles and liquid drops [4, 9] or cells under compression [14] and now as balloons shape. The complete list of Delaunay's surfaces includes cylinders of radius R and mean curvature $H = \dfrac{1}{2R}$, spheres of radius R and mean curvature $H = \dfrac{1}{R}$, catenoids of mean curvature $H = 0$, unduloids of mean curvature $H \neq 0$, and nodoids of mean curvature $H \neq 0$ with profile curves shown at the end in Fig. 1. The analytical description of the three last and most interesting cases from the list above can be found in [6], while in [7] one can find a lot of graphics which are skipped here because of the shortage of space.

Following the plan, we switch to the other example in which the system of equations (17) and (18) can be solved up to the very end. Now we assume that $w(s) = \sigma_c = 0$ and $p(s) = p_o =$ is a non-zero constant. In these circumstances the system formed by (17) and (18) reduces to the equations

$$(23) \qquad (\sigma_m u(s))\frac{\mathrm{d}\theta(s)}{\mathrm{d}s} = -p_o u(s)$$

$$(24) \qquad \frac{\mathrm{d}(\sigma_m u(s))}{\mathrm{d}s} = 0.$$

The last one says that $\sigma_m u(s)$ is a constant as well and allows the previous one to be rewritten as

$$(25) \qquad \frac{\mathrm{d}\theta(s)}{\mathrm{d}s} = -\overset{\circ}{p}\,u(s),$$

where $\overset{\circ}{p}$ is a new constant which can be easily expressed *via* the other constants of the natural shape model. Combined with (3) this equation produces the geometrical relation

$$(26) \qquad u^2(s) = \frac{2}{\overset{\circ}{p}}\cos\theta(s),$$

which, as we shall see, characterizes uniquely the surface in question as follows. Let us solve first (26) for $u(s)$ and after that replace it in (25) in order to get an equation in which the variables are separated

$$(27) \qquad \frac{\mathrm{d}\theta}{\sqrt{\cos\theta}} = -\sqrt{2\overset{\circ}{p}}\,\mathrm{d}s.$$

The later can be easily solved by making use of the standard trigonometric identity $\cos\alpha = 1 - 2\sin^2\alpha/2$ which leads to

$$(28) \qquad \frac{\mathrm{d}(\theta/2)}{\sqrt{1 - 2\sin^2(\theta/2)}} = -\sqrt{\frac{\overset{\circ}{p}}{2}}\,\mathrm{d}s.$$

The integral on the right hand side is trivial while that on the left is just the elliptic integral of the first kind $F(\theta/2, \sqrt{2})$ which can be inverted to produce

$$(29) \qquad \frac{\theta}{2} = \mathrm{am}(-\sqrt{\frac{\overset{\circ}{p}}{2}}s, \sqrt{2})$$

where $\mathrm{am}(t, k)$ is the *Jacobian amplitude function* of the argument t and the *elliptic module* k (for more details on the elliptic functions, their integrals and properties see e.g. [5]). The above mentioned trigonometric identity now gives

$$(30) \qquad \cos\theta = 1 - 2\sin^2\theta/2 = 1 - 2\mathrm{sn}^2(\sqrt{\frac{\overset{\circ}{p}}{2}}s, \sqrt{2}) = \mathrm{dn}^2(\sqrt{\frac{\overset{\circ}{p}}{2}}s, \sqrt{2})$$

and therefore by (26)

$$(31) \qquad u(s) = \sqrt{\frac{2}{\overset{\circ}{p}}\cos\theta} = \sqrt{\frac{2}{\overset{\circ}{p}}}\mathrm{dn}(\sqrt{\frac{\overset{\circ}{p}}{2}}s, \sqrt{2}).$$

Making use of the relation between Jacobian elliptic functions

$$\mathrm{dn}(t/k, k) = \mathrm{cn}(t, 1/k) \tag{32}$$

the result for $u(s)$ can be rewritten finally in the form

$$u(s) = \sqrt{\frac{2}{\mathring{p}}}\,\mathrm{cn}(\sqrt{\mathring{p}}s, \frac{1}{\sqrt{2}}). \tag{33}$$

In order to find $v(s)$ we can use either (4) (in conjuction with the equations (30) and (32)), or the defining arclength equation

$$\frac{\mathrm{d}v(s)}{\mathrm{d}s} = \sqrt{1 - \left(\frac{\mathrm{d}u(s)}{\mathrm{d}s}\right)^2}. \tag{34}$$

Both ways produce the equation

$$\frac{\mathrm{d}v(s)}{\mathrm{d}s} = \mathrm{cn}^2(\sqrt{\mathring{p}}s, \frac{1}{\sqrt{2}}). \tag{35}$$

Details about its integration can be found in [8] and the result is

$$v(s) = \frac{2}{\sqrt{\mathring{p}}}\left[E\left(\mathrm{am}(\sqrt{\mathring{p}}s, \frac{1}{\sqrt{2}}), \frac{1}{\sqrt{2}}\right) - \frac{1}{2}F\left(\mathrm{am}(\sqrt{\mathring{p}}s, \frac{1}{\sqrt{2}}), \frac{1}{\sqrt{2}}\right)\right]. \tag{36}$$

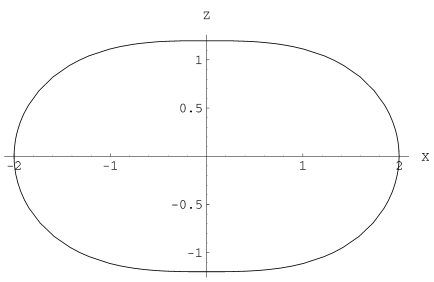

Fig. 2. The profile of the mylar balloon in XOZ plane.

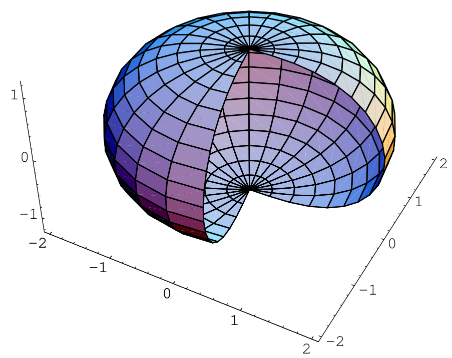

Fig. 3. An open part of the mylar balloon

Comparing the obtained parametrization of the profile curve $(u(s), v(s))$ given by (33) and (36) with one in [8] it is easy to conclude that we are dealing here with the mylar balloon. For commercial purposes the just mentioned mylar balloon is fabricated from two circular disks of mylar, sewing them along their boundaries and then inflating.

Surprisingly enough, these balloons are not spherical as one naively might expect from the well-known fact that the sphere possesses the maximal volume for a given surface area. An experimental fact like this suggests a mathematical problem regarding the exact shape of the balloon when it is fully inflated.

This problem was first spelled out by Paulsen in a variational setting [11] while here we have provided in fact its non-variational characterization. One should mention also the remarkable scale invariance (i.e. independence of the actual size) of the thickness to diameter ratio of the inflated balloon which turns out to be with a good approximation 0.599. Another important fact about this surface is the very simple expression for its area given by the formula $\mathcal{A} = \pi^2 r^2$ where r is the radius of the inflated balloon. Detailed proofs and comments can be found in the above cited papers [6, 7, 8].

REFERENCES

[1] F. Baginski. On the Design and Analysis of Inflated Membranes: Natural and Pumpkin Shaped Balloons. *SIAM J. Appl. Math.*, **65** (2005) 838–857.

[2] F. Baginski, Q. Chen, I. Waldman. Designing the Shape of a Large Scientifc Balloon. *Appl. Math. Modeling*, **25** (2001), 953–966.

[3] C. Delaunay. Sur la surface de revolution dont la courbure moyenne est constante. *J. Math. Pures et Appliquées*, **6** (1841), 309–320.

[4] C. Isenberg. The Science of Soap Films and Soap Bubbles, Dover, New York, 1992.

[5] E. Janhke, F. Emde, F. Ösch. Tafeln Höherer Funktionen, Teubner, Stuttgart, 1960.

[6] I. Mladenov. Delaunay Surfaces Revisited. *C. R. Bulg. Acad. Sci.*, **55** (2002), 19–24.

[7] I. Mladenov, J. Oprea. Unduloids and their Closed Geodesics. In: Proceedings of the Fourth International Conference on Geometry, Integrability and Quantization, Coral Press, Sofia 2003, 206–234.

[8] I. Mladenov, J. Oprea. The Mylar Balloon Revisited. *American Mathematical Monthly*, **110** (2003), 761–784.

[9] J. Oprea. The Mathematics of Soap Films: Explorations with Maple®, AMS, Providence, Rhode Island, 2000.

[10] J. Oprea. Differential Geometry and Its Applications, Second Edition, Prentice Hall, New Jersey, 2004.

[11] W. Paulsen. What is the Shape of the Mylar Balloon? *Amer. Math. Monthly*, **101** (1994), 953–958.

[12] M. Pagitz, Y. Xu, S. Pellegrino. Stability of Lobed Balloons. Cambridge Department of Engineering Preprint, 2005, 21 p.

[13] C. Pozrikidis. Deformed Shapes of Axisymmetric Capsules Enclosed by Elastic Membranes. *J. Eng. Math.*, **45** (2003), 169–182.

[14] M. Yoneda. Tension at the Surface of Sea-Urchin Egg: A Critical Examination of Cole's Expariment. *J. Exp. Biol.*, **41** (1964), 893–906.

[15] Г. Станилов. Диференциална Геометрия. Издателство на С.У., София, 1988.

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БАЛОНИ, МЕМБРАНИ И ПОВЪРХНИНИТЕ СВЪРЗАНИ С ТЯХ

Елена Р. Попова, Мариана Цв. Хаджилазова, Ивайло М. Младенов

Добре известното уравнение на Лаплас-Юнг дава връзката между налягането в отделните фази от двете страни на мембраната или филма и средната кривина на повърхнината, която ги разделя посредством коефициент на пропорционалност носещ името повърхностно напрежение. В настоящата работа са разгледани два случая на споменатото уравнение, които определят повърхнините на Делоне и полиестерният балон и осигуряват тяхното описание, което е независимо от вариационните методи използвани за първоначалното им въвеждане.