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MEASURABILITY OF SETS OF PAIRS OF PARALLEL NON-ISOTROPIC STRAIGHT LINES OF THE SECOND TYPE IN THE SIMPLY ISOTROPIC SPACE*

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We study the measurability of sets of pairs of parallel non-isotropic straight lines in coinciding isotropic planes and the corresponding invariant densities with respect to the group of the general similitudes and some its subgroups.

1. Introduction. The simply isotropic space ${I_3}^{(1)}$ is defined (see [3]) as a projective space $\mathbb{P}_3(\mathbb{R})$ with an absolute consisting of a plane ω (the *absolute plane*) and two complex conjugate straight lines (the *absolute lines*) f_1, f_2 into ω. The absolute lines f_1 and f_2 intersect in a real point F (the *absolute point*). In homogeneous coordinates (x_0, x_1, x_2, x_3) we can choose the plane $x_0 = 0$ as the plane ω, the line $x_0 = 0, x_1 + ix_2 = 0$ as the line f_1 and the line $x_0 = 0, x_1 - ix_2 = 0$ as the line f_2. Then the absolute point F has homogeneous coordinates $(0, 0, 0, 1)$. All regular projectivities transforming the absolute figure into itself form the 8-parametric group G_8 of *the general simply isotropic similitudes.* Passing on to affine coordinates (x, y, z) any similitude of G_8 can be written in the form ([3, p.3])

$$\begin{aligned}\overline{x} &= c_1 + c_7(x\cos\varphi - y\sin\varphi),\\ \overline{y} &= c_2 + c_7(x\sin\varphi + y\cos\varphi),\\ \overline{z} &= c_3 + c_4x + c_5y + c_6z,\end{aligned}$$

where $c_1, c_2, c_3, c_4, c_5, c_6 \neq 0, c_7 > 0$ and φ are real parameters.

A straight line is said to be (completely) *isotropic* if its infinite point coincides with the absolute point F; otherwise, the straight line is said to be *non-isotropic* ([3, p.5]).

We consider G_8 and the following its subgroups:

I. $B_7 \subset G_8 \iff c_7 = 1$. This is the group of the simply isotropic similitudes of the δ-distance ([3, p.5]).

II. $S_7 \subset G_8 \iff c_6 = 1$. This is the group of the simply isotropic similitudes of the s-distance ([3, p.6]).

III. $W_7 \subset G_8 \iff c_6 = c_7$. This is the group of the simply isotropic angular similitudes ([3, p.18]).

IV. $G_7 \subset G_8 \iff \varphi = 0$. This is the group of the simply isotropic boundary similitudes ([3, p.8]).

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V. $V_7 \subset G_8 \iff c_6 c_7^2 = 1$. This is the group of the simply isotropic volume preserving similitudes ([3, p.8]).

VI. $G_6 = G_7 \cap V_7$. This is the group of the simply isotropic volume preserving boundary similitudes ([3, p.8]).

VII. $B_6 = B_7 \cap G_7$. This is the group of the modular boundary motions ([3, p.9]).

VIII. $B_5 = B_7 \cap S_7 \cap G_7$. This is the group of the unimodular boundary motions ([3, p.9]).

Two points P_1 and P_2 are called *parallel* if the straight line P_1P_2 is isotropic.

We emphasize that most of the common material of the geometry of the simply isotropic space $I_3^{(1)}$ can be found in [3].

Using some basic concepts of the integral geometry in the sense of M. I. Stoka [4] and G. I. Drinfel'd [2], we study the measurability of sets of pairs of parallel non-isotropic straight lines in different isotropic planes with respect to G_8 and indicated above subgroups.

2. Measurability with respect to G_8. The pair (G_1, G_2) of parallel non-isotropic straight lines is said to be of the *second type* if G_1 and G_2 lie into an isotropic plane. Assuming that G_1 and G_2 have the equations

$$(1)\qquad \begin{aligned} &G_1 : x = az + p_1, y = bz + q_1, && a \neq 0,\\ &G_2 : x = az + p_2, y = bz + q_1 + \frac{b}{a}\,(p_2 - p_1), && b \neq 0, \end{aligned}$$

we can take the Plücker coordinates ([3; p.38-41]) $g_2^1, g_3^1, g_5^1, g_6^1, g_5^2$ as the parameters of the set of pairs (G_1, G_2), where

$$(2)\qquad g_2^1 = \frac{b}{a},\; g_3^1 = \frac{1}{a},\; g_5^1 = -\frac{p_1}{a},\; g_6^1 = -q_1 + \frac{bp_1}{a},\; g_5^2 = -\frac{p_2}{a}.$$

The associated group $\overline{G}_8$ of G_8 ([3, p.34]) has the infinitesimal operators

$$(3)\qquad \begin{aligned} &Y_1 = g_3^1\Big(\frac{\partial}{\partial g_5^1} + \frac{\partial}{\partial g_5^2}\Big) - g_2^1\frac{\partial}{\partial g_6^1},\;\; Y_2 = \frac{\partial}{\partial g_6^1},\;\; Y_3 = \frac{\partial}{\partial g_5^1} + \frac{\partial}{\partial g_5^2},\;\; Y_4 = \frac{\partial}{\partial g_3^1},\\ &Y_5 = g_2^1\frac{\partial}{\partial g_3^1} - g_6^1\Big(\frac{\partial}{\partial g_5^1} + \frac{\partial}{\partial g_5^2}\Big),\;\; Y_6 = g_3^1\frac{\partial}{\partial g_3^1} + g_5^1\frac{\partial}{\partial g_5^1} + g_5^2\frac{\partial}{\partial g_5^2},\\ &Y_7 = g_3^1\frac{\partial}{\partial g_3^1} - g_6^1\frac{\partial}{\partial g_6^1},\;\; Y_8 = [1 + (g_2^1)^2]\frac{\partial}{\partial g_2^1} + g_2^1\Big(g_3^1\frac{\partial}{\partial g_3^1} + \frac{\partial}{\partial g_6^1}\Big) - g_3^1 g_6^1\Big(\frac{\partial}{\partial g_5^1} + \frac{\partial}{\partial g_5^2}\Big), \end{aligned}$$

and it acts transitively on the set of points $(g_2^1, g_3^1, g_5^1, g_6^1, g_5^2)$. The group $\overline{G}_8$ is isomorphic to G_8 and the invariant density with respect to G_8 of the pairs of lines (G_1, G_2), if it exists, coincides with the density with respect to $\overline{G}_8$ of the points $(g_2^1, g_3^1, g_5^1, g_6^1, g_5^2)$ in the set of parameters. The integral invariant function $f = f(g_2^1, g_3^1, g_5^1, g_6^1, g_5^2)$ of the group G_8 satisfies the system of R. Deltheil ([1, p.28]; [4, p.11])

$$Y_1(f) = 0,\; Y_2(f) = 0,\; Y_3(f) = 0,\; Y_4(f) = 0,\; Y_5(f) = 0,\; Y_6(f) + 3f = 0,$$

$$Y_7(f) = 0,\; Y_8(f) + 4g_2^1 f = 0,$$

and has the form

$$f = \frac{c}{(g_5^2 - g_5^1)^3[1 + (g_2^1)^2]^2},$$

where $c = const$. Thus we are in a position to state the following

Theorem 2.1. *The set of pairs* $(G_1, G_2)(g_2^1, g_3^1, g_5^1, g_6^1, g_5^2)$ *of parallel non-isotropic* *straight lines of the second type is measurable with respect to the group G_8 and has the density*

$$d(G_1, G_2) = \left| \frac{1}{(g_5^2 - g_5^1)^3[1 + (g_2^1)^2]^2} \right| dg_2^1 \wedge dg_3^1 \wedge dg_5^1 \wedge dg_6^1 \wedge dg_5^2. \tag{4}$$

Differentiating (2) and substituting into (4) we obtain another expression for the density:

Corollary 2.1. *The density* (4) *for the pairs* (G_1, G_2)*, determined by the equations* (1)*, can be written in the form*

$$d(G_1, G_2) = \left| \frac{a^2}{(p_2 - p_1)^3(a^2 + b^2)^2} \right| da \wedge db \wedge dp_1 \wedge dq_1 \wedge dp_2. \tag{5}$$

3. Measurability with respect to S_7. The associated group $\overline{S}_7$ of the group S_7 has the infinitesimal operators $Y_1, Y_2, Y_3, Y_4, Y_5, Y_7$, and Y_8 from (3). Since $\overline{G}_7$ acts intransitively on the set of points $(g_2^1, g_3^1, g_5^1, g_6^1, g_5^2)$, the set of pairs (G_1, G_2) is not measurable with respect to G_7.

The system $Y_i(f) = 0,\ i = 1, \dots, 7$, has a solution $f = g_5^2 - g_5^1$, and it is an absolute invariant of $\overline{S}_7$.

Consider the subset of pairs (G_1, G_2) satisfying the condition

$$g_5^2 - g_5^1 = 0, \tag{6}$$

where h=const. The group $\overline{S}_7$ induces the group $S_7^\star$ on the subset (6) with the infinitesimal operators

$$Y_1^\star = g_3^1 \frac{\partial}{\partial g_5^1} - g_2^1 \frac{\partial}{\partial g_6^1},\ Y_2 = \frac{\partial}{\partial g_6^1},\ Y_3^\star = \frac{\partial}{\partial g_5^1},\ Y_4 = \frac{\partial}{\partial g_3^1},\ Y_5^\star = g_2^1 \frac{\partial}{\partial g_3^1} - g_6^1 \frac{\partial}{\partial g_5^1},$$
$$Y_7 = g_3^1 \frac{\partial}{\partial g_3^1} - g_6^1 \frac{\partial}{\partial g_6^1},\ Y_8^\star = = [1 + (g_2^1)^2] \frac{\partial}{\partial g_2^1} + g_2^1 (g_3^1 \frac{\partial}{\partial g_3^1} + g_6^1 \frac{\partial}{\partial g_6^1}) - g_3^1 g_6^1 \frac{\partial}{\partial g_5^1},$$

and it is transitive. The Deltheil system $Y_1^\star = 0$, $Y_2(f) = 0$, $Y_3^\star(f) = 0$, $Y_4(f) = 0$, $Y_5^\star(f) = 0$, $Y_7(f) = 0$, $Y_8^\star(f) + 4g_2^1 f = 0$ has the solution

$$f = \frac{c}{[1 + (g_2^1)^2]^2},$$

where $c = const$.

From here it follows:

Theorem 3.1. *The set of pairs* $(G_1, G_2)(g_2^1, g_3^1, g_5^1, g_6^1, g_5^2)$ *of parallel non-isotropic straight lines of the second type is not measurable with respect to the group S_7, but it has the measurable subset* (6) *with the density*

$$g(G_1, G_2) = \left| \frac{c}{[1 + (g_2^1)^2]^2} \right| dg_2^1 \wedge dg_3^1 \wedge dg_5^1 \wedge dg_6^1.$$

From Theorem 3.1. and (2) it follows:

Corollary 3.1. *The set of pairs* $(G_1, G_2)(a, b, p_1, q_1, p_2)$ *of parallel non-isotropic straight lines of the second type is not measurable with respect to the group S_7, but it* *has the measurable subset* $\dfrac{p_1 - p_2}{a} = h$, h=const, *with the density*

$$d(G_1, G_2) = \frac{1}{(a^2 + b^2)^2}\, da \wedge db \wedge dp_1 \wedge dq_1.$$

4. Measurability with respect to B_7, W_7, G_7, V_7, G_6, B_6, B_5. By arguments similar to the ones used above we study the measurability of sets of pairs (G_1, G_2) with respect to all the remaining groups. We summarize the results in the following

Theorem 4.1. *The set of pairs* $(G_1, G_2)(g_2^1, g_3^1, g_5^1, g_6^1, g_5^2)$ *of parallel non-isotropic straight lines of the second type:*

(i) *is measurable with respect to the group* B_7 *and* W_7, *and has the density* (4)*;*

(ii) *is measurable with respect to the group* V_7, *and has the density*

$$d(G_1, G_2) = \left|\frac{1}{(g_5^1 - g_5^2)^5[1 + (g_2^1)^2]^2}\right| dg_2^1 \wedge dg_3^1 \wedge dg_5^1 \wedge dg_6^1 \wedge dg_5^2;$$

(iii) *is not measurable with respect to the groups* G_7, G_6, *and* B_6, *but it has the measurable subset* $g_2^1 = h, h = const$, *with the density*

$$d(G_1, G_2) = \left|\frac{1}{(g_5^2 - g_5^1)^3}\right| dg_3^1 \wedge dg_5^1 \wedge dg_6^1 \wedge dg_5^2.$$

(iv) *is not measurable with respect to the group* B_5, *but it has the measurable subset* $g_2^1 = h_1, g_5^1 - g_5^2 = h_2, h_1 = \text{const}, h_2 = \text{const}$, *with the density*

$$d(G_1, G_2) = dg_3^1 \wedge dg_5^1 \wedge dg_6^1.$$

From Theorem 4.1. and (4) it follows

Corollary 4.1. *The set of pairs* $(G_1, G_2)(a, b, p_1, q_1, p_2)$ *of parallel non-isotropic straight lines of the second type:*

(i) *is measurable with respect to the group* B_7, W_7 *and has the density* (5)*;*

(ii) *is measurable with respect to the group* V_7, *and has the density*

$$d(G_1, G_2) = \left|\frac{a^4}{(p_2 - p_1)^5(a^2 + b^2)^2}\right| da \wedge db \wedge dp_1 \wedge dq_1 \wedge dp_2;$$

(iii) *is not measurable with respect to the group* G_7, G_6, B_5, *but it has the measurable subset* $\frac{b}{a} = h, h = const$, *with the density*

$$d(G_1, G_2) = \left|\frac{1}{a(p_2 - p_1)^3}\right| da \wedge dp_1 \wedge dq_1 \wedge dp_2;$$

(iv) *is not measurable with respect to the group* B_5, *but it has the measurable subset* $\dfrac{b}{a} = h_1, \dfrac{p_2 - p_1}{a} = h_2, h_1 = const, h_2 = const$, *with the density*

$$d(G_1, G_2) = \frac{1}{|a|^3}\, da \wedge dp_1 \wedge dq_1.$$

5. Some Crofton type formulas with respect to G_8. The oriented s-distance from G_1 to G_2 if

$$s = \frac{p_2 - p_1}{a}. \tag{7}$$

Denoting $\overline{P}_1 = G_1 \cap Oxy$, $\overline{P}_2 = G_2 \cap Oxy$, and $\theta = \sphericalangle(G_1, Oxy) = \sphericalangle(G_2, Oxy)$ we have $\overline{P}_1(p_1, q_1, 0)$, $\overline{P}_2(p_2, q_+ \dfrac{b}{a}\,(p_2 - p_1), 0)$ and [3; p. 48]

$$\theta = \frac{1}{\sqrt{a^2+b^2}}. \tag{8}$$

We compute

$$da \wedge db \wedge dp_1 \wedge dq_1 \wedge dp_2 = \frac{a}{s^2}\, ds \wedge d\overline{P}_1 \wedge d\overline{P}_2, \tag{9}$$

$$da \wedge db \wedge dp_1 \wedge dq_1 \wedge dp_2 = -\frac{a}{s\theta}\, d\theta \wedge d\overline{P}_1 \wedge d\overline{P}_2, \tag{10}$$

where $d\overline{P}_i$ is the density of the points $\overline{P}_i$ into the Euclidean plane Oxy.

Substituting (7), (8), (9), and (10) into (5), we find

$$d(G_1, G_2) = \frac{\theta^4}{|s|^5}\, ds \wedge d\overline{P}_1 \wedge d\overline{P}_2, \tag{11}$$

$$d(G_1, G_2) = \frac{\theta^3}{s^4}\, d\theta \wedge d\overline{P}_1 \wedge d\overline{P}_2, \tag{12}$$

respectively. Thus we have the following

Theorem 5.1. *The density for the pairs* (G_1, G_2) *of parallel non-isotropic straight lines* (1) *with respect to the group* G_8 *satisfies the relations* (11) *and* (12)*.*

REFERENCES

[1] R. Deltheil. Sur la théorie des probabilité géométriques. *Thése Ann. Fac. Sc. Univ. Toulouse*, (3), **11** (1919), 1–65.

[2] G. I. Drinfel'd. On the measure of the Lie groups. *Zap. Mat. Otdel. Fiz. Mat. Fak. Kharkov. Mat. Obsc.*, **21** (1949), 47–57 (in Russian).

[3] H. Sachs. Isotrope Geometrie des Raumes. Friedr. Vieweg and Sohn, Braunschweig/Wiesbaden, 1990.

[4] M. I. Stoka. Geometrie Integrală. Ed. Acad. RPR, Bucuresti, 1967.

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ИЗМЕРИМОСТ НА МНОЖЕСТВА ОТ ДВОЙКИ УСПОРЕДНИ НЕИЗОТРОПНИ ПРАВИ ОТ ВТОРИ ТИП В ПРОСТО ИЗОТРОПНО ПРОСРТРАНСТВО

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Изследвана е измеримостта на множества от двойки успоредни неизотропни прави, лежащи в една и съща изотропна равнина и съответните инвариантни гъстоти относно групата на общите просто-изотропни подобности и някои нейни подгрупи.