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CURVATURE PROPERTIES OF CONFORMAL KÄHLER MANIFOLDS WITH NORDEN METRIC*

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The class of the manifolds which are conformally equivalent to the Kähler manifolds with Norden metric is considered. The curvature tensor on such four-dimensional manifolds is obtained. The case of isotropic Kähler manifolds with Norden metric is studied. A transformation of the Levi-Civita connections of two Norden metrics is considered. Some invariant tensors of this transformation are obtained.

1. Preliminaries. Let (M,J,g) be a $2n$-dimensional almost complex manifold with Norden metric, i.e. J is an almost complex structure and g is a metric on M such that:

$$J^2X=-X,\qquad g(JX,JY)=-g(X,Y),\qquad X,Y\in\mathfrak{X}(M). \tag{1.1}$$

The associated metric $\widetilde{g}$ of g on M given by $\widetilde{g}(X,Y)=g(X,JY)$ is a Norden metric too. Both metrics are necessarily of signature (n,n).

Further, X,Y,Z,W $(x,y,z,w$, respectively) will stand for arbitrary differentiable vector fields on M (vectors in T_pM, $p\in M$, respectively).

Let ∇ be the Levi-Civita connection of the metric g. Then, the tensor field F of type $(0,3)$ on M is defined by

$$F(X,Y,Z)=g\left((\nabla_X J)Y,Z\right). \tag{1.2}$$

It has the following symmetries

$$F(X,Y,Z)=F(X,Z,Y)=F(X,JY,JZ). \tag{1.3}$$

Let $\{e_i\}$ $(i=1,2,\ldots,2n)$ be an arbitrary basis of T_pM at a point p of M. The components of the inverse matrix of g are denoted by g^{ij} with respect to the basis $\{e_i\}$.

The Lie form θ associated with F is defined by

$$\theta(z)=g^{ij}F(e_i,e_j,z) \tag{1.4}$$

and the corresponding Lie vector is denoted by Ω, i.e. $\theta(z)=g(z,\Omega)$.

A classification of the considered manifolds with respect to the tensor F is given in [1]. Eight classes of almost complex manifolds with Norden metric are characterized there according to the properties of F. The three basic classes W_1, W_2, W_3 and the class $W_1\oplus W_2$ of the complex manifolds with Norden metric are given as follows:

$$\begin{aligned}W_1: F(X,Y,Z)=\frac{1}{2n}&\left[g(X,Y)\theta(Z)+g(X,Z)\theta(Y)\right.\\&\left.+g(X,JY)\theta(JZ)+g(X,JZ)\theta(JY)\right];\end{aligned} \tag{1.5}$$

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$$W_2 : F(X,Y,JZ) + F(Y,Z,JX) + F(Z,X,JY) = 0, \quad \theta = 0;$$

$$W_3 : F(X,Y,Z) + F(Y,Z,X) + F(Z,X,Y) = 0;$$

$$W_1 \oplus W_2 : F(X,Y,JZ) + F(Y,Z,JX) + F(Z,X,JY) = 0. \tag{1.6}$$

The special class W_0 of the Kähler manifolds with Norden metric belonging to any other class is determined by the condition $F = 0$.

Let R be the curvature tensor of ∇, i.e.

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{1.7}$$

The corresponding tensor of type $(0,4)$ is denoted by the same letter and is given by $R(X,Y,Z,W) = g\left(R(X,Y)Z,W\right)$.

A tensor L of type $(0,4)$ is called *a curvature-like tensor* if it satisfies the following conditions for any $X,Y,Z,W \in \mathfrak{X}(M)$:

$$L(X,Y,Z,W) = -L(Y,X,Z,W) = -L(X,Y,W,Z),$$

$$L(X,Y,Z,W) + L(Y,Z,X,W) + L(Z,X,Y,W) = 0.$$

Then, the Ricci tensor $\rho(L)$ and the scalar curvatures $\tau(L)$ and $\tau^*(L)$ of L are defined by:

$$\rho(L)(y,z) = g^{ij} L(e_i,y,z,e_j); \; \tau(L) = g^{ij}\rho(L)(e_i,e_j); \; \tau^*(L) = g^{ij}\rho(L)(e_i,Je_j). \tag{1.8}$$

A curvature-like tensor L is called *a Kähler tensor* if it satisfies the condition

$$L(X,Y,JZ,JW) = -L(X,Y,Z,W), \qquad X,Y,Z,W \in \mathfrak{X}(M). \tag{1.9}$$

Further, let S be a symmetric and hybrid with respect to J tensor of type $(0,2)$, i.e. $S(JX,Y) = S(JY,X)$. We consider the following curvature-like tensors of type $(0,4)$:

$$\begin{aligned} \psi_1(S)(X,Y,Z,W) &= g(Y,Z)S(X,W) - g(X,Z)S(Y,W) \\ &\quad + g(X,W)S(Y,Z) - g(Y,W)S(X,Z); \\ \psi_2(S)(X,Y,Z,W) &= g(Y,JZ)S(X,JW) - g(X,JZ)S(Y,JW) \\ &\quad + g(X,JW)S(Y,JZ) - g(Y,JW)S(X,JZ); \end{aligned} \tag{1.10}$$

$$\pi_1 = \frac{1}{2}\psi_1(g); \qquad \pi_2 = \frac{1}{2}\psi_2(g); \qquad \pi_3 = -\psi_1(\widetilde{g}) = \psi_2(\widetilde{g}).$$

It is well known that the Weyl tensor W on a $2n$-dimensional pseudo-Riemannian manifold $(n \geq 2)$ is defined as follows

$$W = R - \frac{1}{2n-2}\left\{\psi_1(\rho) - \frac{\tau}{2n-1}\pi_1\right\}. \tag{1.11}$$

Let $\alpha = \{x,y\}$ be a non-degenerate 2-plane spanned by vectors $x, y \in T_pM$, $p \in M$. The sectional curvatures of α with respect to the curvature-like tensor L are given by

$$\nu(L;p) = \frac{L(x,y,y,x)}{\pi_1(x,y,y,x)}, \qquad \nu^*(L;p) = \frac{L(x,y,y,Jx)}{\pi_1(x,y,y,x)}. \tag{1.12}$$

The square norm $\|\nabla J\|^2$ of ∇J is defined in [3] by

$$\|\nabla J\|^2 = g^{ij}g^{kl}g\left((\nabla_{e_i}J)e_k, (\nabla_{e_j}J)e_l\right). \tag{1.13}$$

Following [3], [4] we define a second square norm $\|\nabla J\|_*^2$ of ∇J with respect to the associated metric $\widetilde{g}$ by

$$\|\nabla J\|_*^2 = \widetilde{g}^{ij}\widetilde{g}^{kl}\widetilde{g}\left((\nabla_{e_i}J)e_k, (\nabla_{e_j}J)e_l\right), \tag{1.14}$$

where $\widetilde{g}^{ij} = -J_s^i g^{js}$ are the components of the inverse matrix of $\widetilde{g}$ with respect to the basis $\{e_i\}$. Then, having in mind the definition (1.2) and the properties (1.3) of the tensor F, from (1.13) and (1.14) we obtain that

$$\|\nabla J\|^2 = g^{ij}g^{kl}g^{pq}F_{ikp}F_{jlq}; \qquad \|\nabla J\|_*^2 = -\widetilde{g}^{ij}g^{kl}g^{pq}F_{ikp}F_{jlq}, \tag{1.15}$$

where $F_{ikp} = F(e_i, e_k, e_p)$.

Definition 1.1. *An almost complex manifold with Norden metric satisfying the condition $\|\nabla J\|^2 = 0$ is called an isotropic Kähler manifold with Norden metric.*

Definition 1.2. *An almost complex manifold with Norden metric satisfying the condition $\|\nabla J\|^2 = \|\nabla J\|_*^2 = 0$ is called a strong isotropic Kähler manifold with Norden metric.*

2. Complex connections and curvature tensors on conformal Kähler manifolds with Norden metric. Let (M, J, g) be a W_1-manifold with Norden metric. The Lie forms θ and $\theta^* = \theta \circ J$ are closed on M if and only if $(\nabla_X\theta)\,Y = (\nabla_Y\theta)\,X$ and $(\nabla_X\theta)\,JY = (\nabla_Y\theta)\,JX$. A W_1-manifold with closed Lie forms is called *a conformal Kähler manifold with Norden metric*. The subclass of these manifolds is denoted by W_1^0.

In [2] is introduced a cannonical linear connection (so called B-connection) D on a complex manifold with Norden metric as follows

$$D_XY = \nabla_XY - \frac{1}{2}J\,(\nabla_XJ)\,Y. \tag{2.1}$$

It is shown that g and J are parallel with respect to the connection D. The curvature tensor K of D is proved to be Kählerian.

In [6] is studied the Yano connection ∇' given by

$$\nabla'_XY = \nabla_XY + \frac{1}{4}\left\{(\nabla_XJ)\,JY + 2\,(\nabla_YJ)\,JX - (\nabla_{JX}J)\,Y\right\}.$$

It is proved that the Yano connection is torsion-free and that $\nabla'J = 0$ on a complex manifold with Norden metric. In the same paper is obtained the Kähler curvature tensor R' of type $(0,4)$ of ∇' on a W_1^0-manifold as follows

$$R' = R - \frac{1}{4n}\left\{\psi_1 + \psi_2\right\}(S) - \frac{1}{8n^2}\psi_1(M) - \frac{\theta(\Omega)}{16n^2}\left\{3\pi_1 + \pi_2\right\} + \frac{\theta(J\Omega)}{16n^2}\pi_3, \tag{2.2}$$

where

$$\begin{aligned} S(X,Y) &= (\nabla_X\theta)\,JY + \frac{1}{4n}\left[\theta(X)\theta(Y) - \theta(JX)\theta(JY)\right], \\ M(X,Y) &= \theta(X)\theta(Y) + \theta(JX)\theta(JY). \end{aligned} \tag{2.3}$$

Then, having in mind (1.7), (2.1), (2.2) and (2.3) we obtain:

Theorem 2.1. *The Kähler curvature tensors of the connections D and ∇' coincide on a conformal Kähler manifold with Norden metric, i.e. $K = R'$.*

Theorem 2.2. *Let (M, J, g) be a four-dimensional almost complex manifold with Norden metric and L be a Kähler tensor on M. Then, the tensor L has the following form*

$$L = \nu(L)\left\{\pi_1 - \pi_2\right\} + \nu^*(L)\pi_3. \tag{2.4}$$

Proof. It is known [5] that in the tangent space T_pM, $p \in M$, there exists a J-basis $\{e_1, e_2, Je_1, Je_2\}$ such that $g(e_i, e_j) = -g(Je_i, Je_j) = \delta_{ij}$, $g(e_i, Je_j) = 0$, $i, j = 1, 2$. Then, by the use of (1.9), (1.10), (1.12) and after straightforward calculations we prove the truthfulness of (2.4). $\square$

From the last theorem and (1.8) it follows that

$$\nu(L) = \frac{\tau(L)}{8}, \qquad \nu^*(L) = \frac{\tau^*(L)}{8}. \tag{2.5}$$

Then, having in mind (2.2) and (2.3) for $n = 2$, (2.4), (2.5) and (1.8) we obtain the following

Theorem 2.3. *Let (M, J, g) be a four-dimensional W_1^0-manifold. Then, for the curvature tensor R of the Levi-Civita connection ∇ we have*

$$\begin{aligned} R = {} & \frac{\tau - \operatorname{div}(J\Omega)}{8}\{\pi_1 - \pi_2\} + \frac{\operatorname{tr} S^*}{16}\pi_3 - \frac{1}{8}\{\psi_1 - \psi_2\}(S) + \frac{1}{2}\left\{\psi_1(\rho) - \frac{\tau}{3}\pi_1\right\} \\ & + \frac{1}{4}\left[\frac{\operatorname{div}(J\Omega)}{2} - \frac{\tau}{3} - \frac{\theta(\Omega)}{8}\right]\pi_1, \end{aligned} \tag{2.6}$$

where $\operatorname{tr} S^* = g^{ij} S(e_i, Je_j) = -\operatorname{div}\Omega + \dfrac{\theta(J\Omega)}{4}$ *for* $n = 2$, $\operatorname{div}\Omega = \nabla_i \Omega^i$ *and* $\operatorname{div}(J\Omega) = \nabla_i(J_k^i \Omega^k)$.

The last theorem and (1.11) imply the following

Corollary 2.1. *Let (M, J, g) be four-dimensional W_1^0-manifold. Then, for the Weyl tensor we have*

$$W = \frac{\tau - \operatorname{div}(J\Omega)}{8}\{\pi_1 - \pi_2\} + \frac{\operatorname{tr} S^*}{16}\pi_3 - \frac{1}{8}\{\psi_1 - \psi_2\}(S) + \frac{1}{4}\left[\frac{\operatorname{div}(J\Omega)}{2} - \frac{\tau}{3} - \frac{\theta(\Omega)}{8}\right]\pi_1.$$

Next, taking into account (1.5) and (1.15) we obtain that on a W_1-manifold

$$\|\nabla J\|^2 = \frac{\theta(\Omega)}{n^2}, \qquad \|\nabla J\|_*^2 = -\frac{\theta(J\Omega)}{n^2}, \tag{2.7}$$

and by the use of (1.3), (1.5), (1.6) and (2.7) we obtain the following

Lemma 2.1. *Let (M, J, g) be an isotropic Kähler W_1-manifold with Norden metric. Then, M is a strongly isotropic Kähler W_1-manifold with Norden metric.*

Now, (2.6) and the last lemma imply the following

Corollary 2.2. *Let (M, J, g) be a four-dimensional isotropic Kähler W_1^0-manifold. Then, for the curvature tensor R of ∇ we have*

$$\begin{aligned} R = {} & \frac{\tau - \operatorname{div}(J\Omega)}{8}\{\pi_1 - \pi_2\} - \frac{\operatorname{div}\Omega}{16}\pi_3 - \frac{1}{8}\{\psi_1 - \psi_2\}(S) + \frac{1}{2}\left\{\psi_1(\rho) - \frac{\tau}{3}\pi_1\right\} \\ & + \frac{1}{4}\left[\frac{\operatorname{div}(J\Omega)}{2} - \frac{\tau}{3}\right]\pi_1. \end{aligned}$$

3. Invariant tensors of the transformation of the Levi-Civita connections of g and $\widetilde{g}$ on a W_1-manifold. Let (M, J, g) be an almost complex manifold with Norden metric and $\widetilde{\nabla}$ be the Levi-Civita connection of the associated metric $\widetilde{g}$. In [2] is considered the tensor

$$\Phi(X,Y,Z)=g\left(\widetilde{\nabla}_X Y-\nabla_X Y,Z\right)$$

and it is obtained that

$$\Phi(X,Y,Z)=\frac{1}{2}\left\{F(JZ,X,Y)-F(X,Y,JZ)-F(Y,X,JZ)\right\}. \tag{3.1}$$

By the use of (1.5) and (3.1) we receive the following

Lemma 3.1. *Let (M,J,g) be a W_1-manifold with Norden metric. Then, for the connections ∇ and $\widetilde{\nabla}$ we have*

$$\widetilde{\nabla}_X Y=\nabla_X Y+\frac{1}{2n}\left[g(X,JY)\Omega-g(X,Y)J\Omega\right]. \tag{3.2}$$

Let $\widetilde{R}$ be the curvature tensor of $\widetilde{\nabla}$. Then, having in mind (1.7) and (3.2) we obtain

Theorem 3.1. *Let (M,J,g) be a W_1-manifold with Norden metric. Then, the curvature tensors R and $\widetilde{R}$ of type $(1,3)$ are related as follows*

$$\begin{aligned}\widetilde{R}(X,Y)Z=R(X,Y)Z+\frac{1}{2n}\Big\{&g(X,Z)\left[\nabla_Y J\Omega-\frac{1}{2n}\theta(JY)J\Omega\right]\\
&-g(Y,Z)\left[\nabla_X J\Omega-\frac{1}{2n}\theta(JX)J\Omega\right]-g(X,JZ)\left[\nabla_Y\Omega-\frac{1}{2n}\theta(Y)J\Omega\right]\\
&+g(Y,JZ)\left[\nabla_X\Omega-\frac{1}{2n}\theta(X)J\Omega\right]\Big\}.\end{aligned} \tag{3.3}$$

Further, we consider the following tensors:

$$\begin{aligned}T_1(X,Y)Z&=R(X,Y)Z+\frac{1}{4n}\Big\{g(X,Z)\left[\nabla_Y J\Omega-\frac{1}{2n}\theta(JY)J\Omega\right]\\
&\quad-g(Y,Z)\left[\nabla_X J\Omega-\frac{1}{2n}\theta(JX)J\Omega\right]-g(X,JZ)\left[\nabla_Y\Omega-\frac{1}{2n}\theta(Y)J\Omega\right]\\
&\quad+g(Y,JZ)\left[\nabla_X\Omega-\frac{1}{2n}\theta(X)J\Omega\right]\Big\};\\
T_2(X,Y)&=\rho(X,Y)-\frac{1}{2n}\left[g(X,Y)\tau-g(X,JY)\tau^*\right];\\
T_3(X,Y)&=\left(\nabla_X\theta\right)Y+\frac{1}{4n}\left[g(X,Y)\theta(J\Omega)-g(X,JY)\theta(\Omega)\right].\end{aligned}$$

Then, by the use of (1.2), (1.4), (3.2), (3.3) and $\left(\nabla_X\theta\right)Y=X\theta(Y)-\theta(\nabla_X Y)$ we get the following

Theorem 3.2. *Let (M,J,g) be a W_1-manifold with Norden metric. Then, the Lie form θ and the tensors T_1, T_2, T_3 are invariant by the transformation of the connections ∇ and $\widetilde{\nabla}$.*

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КРИВИННИ СВОЙСТВА НА КОНФОРМНО КЕЛЕРОВИ МНОГООБРАЗИЯ С НОРДЕНОВА МЕТРИКА

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Разгледан е класът на многообразията, конформно еквивалентни с келерови многообразия с норденова метрика. Намерен е видът на тензора на кривината върху такива четиримерни многообразия. Изследван е случаят на изотропно келерови многообразия с норденова метрика. Разгледано е преобразуването на свързаностите на Леви-Чивита на норденовите метрики. Намерени са някои инвариантни тензори на тази трансформация